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Categorical-algebraic methods in non-commutative and non-associative algebra

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Categorical-algebraic methods in non-commutative and non-associative algebra

AUTORIZACIÓN DEL DIRECTOR/TUTOR DE LA TESIS

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Contents

Introduction

Ever since the introduction of abelian categories in the 1950s as a categorical abstraction of the properties of abelian groups and modules, there has been the wish to find a similar framework that nicely reflects the properties of (not necessarily abelian) groups, rings and algebras.

Over the years, several attempts at such an abstract framework have been made: worth mentioning here is the work of Higgins [24], Huq [26] and R.-Grandjeán [32], see [27] for an exhaustive list. Since none of these attempts were entirely successful, and furthermore the connections between them were not clear, some of these approaches were not further developed or even given a name. In 1999, Janelidze, Márki and Tholen realised that *Barr-exactness* [2], combined with the concept of *Bourn-protomodularity* [5], provides a context which simplifies and unifies the above-mentioned "old" axiom systems, and in which the relationships with modern categorical algebra can be explored.

Expressed in terms of "new" axioms, a *semi-abelian category* is a pointed category which is *Barr-exact* and *Bourn-protomodular* with finite sums. Examples of semi-abelian categories are abundant and ubiquitous. In particular, we may find many of the non-associative and non-commutative algebraic structures studied in the literature [3], including all those that have an underlying group structure. More precisely, any pointed variety of algebras which has amongst its operations and identities those of the theory of groups is semi-abelian.

One of the advantages of this categorical framework is that it allows a unified study of many important homological properties. For instance, in any semi-abelian category, the classical diagram lemmas (the *Short Five Lemma*, the 3×3 *Lemma*, the *Snake Lemma*, *Noether's Isomorphism Theorems*) hold. As seen in [34], the theory of semi-abelian categories is perfectly suited for the study of non-abelian (co)homology and the corresponding homotopy theory, unifying many basic aspects of the classical (co)homology theories of groups, Lie algebras and crossed modules.

From an algebraic point of view, the cohomology theory of Lie algebras was introduced in [11], aiming to give an algebraic construction of the cohomology of topological spaces of compact Lie groups. It was very much studied through the years and extended to other related structures, such as *crossed modules of Lie algebras* [8, 7], *Lie superalgebras* [31], *Lie-Rinehart algebras* [25, 33], *Leibniz (super)algebras* [28], *n-Lie algebras* [1], *n-Leibniz algebras* [9], etc.

The theory of non-associative algebras is strongly related to different areas of mathematics and it has many applications in physics, mechanics, biology and other sciences. Foremost amongst them are the theories of Lie and Jordan algebras, which have had an enormous relevance in the past century. The study of non-associative algebras encompasses the theory of not necessarily associative R-algebras (with associative algebras being an important special case), where R may be a ring or a field. The problems arising in these topics are of various kinds, such as the study of solvability and nilpotency, classifications, characterisations, relations with differential geometry and manifolds, etc.

The objective of this dissertation is twofold: firstly to use categorical and algebraic methods to study homological properties of some of the aforementioned semi-abelian, non-associative structures and secondly to use categorical and algebraic methods to study categorical properties and provide categorical characterisations of some well-known algebraic structures. On one hand, the theory of universal central extensions together with the non-abelian tensor product will be studied and used to explicitly calculate some homology groups [10, 12, 17, 19, 18] and some problems about universal enveloping algebras and actions will be solved [14, 15, 6, 20]. On the other hand, we will focus on giving categorical characterisations of some algebraic structures, such as a characterisation of groups amongst monoids [16], of cocommutative Hopf algebras amongst cocommutative bialgebras [22] and of Lie algebras amongst alternating algebras [21].

Since each chapter will have an explicit and fully detailed introduction it does not seem necessary to overextend this first general overview. It is worth mentioning here that notations might not be coherent throughout the text, nevertheless the notation in each chapter is fully internally consistent.

The dissertation is organised as follows: In Chapter 1, the universal central extension of a Lie-Rinehart algebra is described and related with the generalisation of Ellis's *non-abelian tensor product* [13] of Lie algebras to Lie-Rinehart algebras. Chapter 2 is devoted to introducing the non-abelian tensor product of Lie superalgebras, relating it with universal central extensions. Here also the low-dimensional *non-abelian homology* is introduced and its relationship with the cyclic homology of associative superalgebras is established. In Chapter 3 an explicit computation of $\mathrm{H}_2\big(\mathfrak{sl}(m,n,A)\big)$ and $\mathrm{H}_2\big(\mathfrak{st}(m,n,A)\big)$ is obtained, where A is a superalgebra and $3 \leqslant m+n \leqslant 5$, and connections with the cyclic homology of associative superalgebras are made. Later on, these results are extended in Chapter 4 to the case of superdialgebras for any $m+n \geqslant 3$, with the additional interest of introducing a new method using the non-abelian tensor product. A generalisation of Ellis's non-abelian exterior product [13] to Leibniz algebras is given in Chapter 5, where it is applied to the construction of an eight term exact sequence in Leibniz homology.

Chapter 6 is devoted to extending the notion of *biderivation* of Leibniz algebras to the crossed modules setting, and to check in which situations it behaves as the *actor* of the category (also called the *split extension classifier* in [4]). In Chapter 7, the universal enveloping algebra of a crossed module of Leibniz algebras is studied using new techniques. Then, it is seen as a particular case of crossed modules of Lie algebras in the *Loday-Pirashvili* category [29]. In Chapter 8 a proper definition of the universal enveloping algebra functor of n-Lie algebras is given, and it is proven that this functor cannot have a right adjoint.

In Chapter 9 there is a sharpened version of the characterisation of groups amongst monoids given by Montoli, Rodelo and Van der Linden in [30], proving that a monoid is a group if and only if all splits extensions over it are strong. This characterisation is generalised in Chapter 10, where the following result is obtained: a cocommutative bialgebra over an algebraically closed field is a Hopf algebra if and only if all splits extensions over it are stably strong. In this chapter it is also shown that the category of (not necessarily commutative or cocommutative) Hopf algebras is not unital, so in particular is not semi-abelian. Using Gray's notion of *locally algebraically cartesian closed* category [23], in Chapter 11 a characterisation of Lie algebras amongst all varieties of non-associative, alternating algebras is given. The result is a categorical characterisation of the Jacobi identity.

Finally, Chapter 12 is devoted to saying some words about the current state of affairs of several works in progress that are taking shape right now, and to mention some lines of research that may be followed when taking this dissertation as a starting point.

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Chapter 1

Universal central extensions of Lie-Rinehart algebras

Abstract

In this paper we study the universal central extension of a Lie-Rinehart algebra and we give a description of it. Then we study the lifting of automorphisms and derivations to central extensions. We also give a definition of a non-abelian tensor product in Lie-Rinehart algebras based on the construction of Ellis of non-abelian tensor product of Lie algebras. We relate this non-abelian tensor product to the universal central extension.

Reference

J. L. Castiglioni, X. García-Martínez, and M. Ladra, *Universal central extensions of Lie-Rinehart algebras*, J. Algebra Appl., 2017, doi:10.1142/S0219498818501347.

1.1 Introduction

Let A be a unital commutative algebra over a commutative ring K with unit. A Lie-Rinehart algebra is a Lie K-algebra, which is also an A-module and these two structures are related in an appropriate way [11]. The leading example of Lie-Rinehart algebras is the set $\mathtt{Der}_K(\mathrm{A})$ of all K-derivations of A. Lie-Rinehart algebras are the algebraic counterpart of Lie algebroids [20].

The concept of Lie-Rinehart algebra generalizes the notion of Lie algebra. In [24] and [8] universal central extensions of Lie algebra are studied, proving that if a Lie algebra is perfect then it has a universal central extension. Moreover, it is characterized the kernel of the universal central extension as the second homology group with trivial coefficients. In this paper we extend this study to Lie-Rinehart algebras.

On the other hand, in [6] a non-abelian tensor product of Lie algebras is introduced, its more important properties are studied and it is related to the universal central extension. It has been extended to other structures as Leibniz algebras [9], Lie superalgebras [7] or Hom-Lie algebras [3]. In this paper we broaden this construction to Lie-Rinehart algebras, we study some important properties and we relate it to the universal central extension of Lie-Rinehart algebras.

After the introduction, the paper is organized in four sections. In Sec. 1.2, we recall some needed notions and facts on Lie-Rinehart algebras, actions, crossed modules, universal enveloping algebras, free algebras, homology and cohomology and abelian extensions of Lie-Rinehart algebras. In Sec. 1.3, following Neher's paper on Lie superalgebras [21], we introduce central extensions and universal central extensions of Lie-Rinehart algebras giving a characterization of them (Theorem 1.3.7), extending classic results of Lie algebras (see [24]). We construct an endofunctor $\mathfrak{uce}_{\mathrm{A}}$ that when the Lie-Rinehart algebra is perfect gives explicitly the universal central extension. In Sec. 1.4, we study the lifting of automorphisms and derivations to central extensions. Finally, in Sec. 1.5, we introduce a non-abelian tensor product of Lie-Rinehart algebras extending Ellis [6] non-abelian tensor product of Lie algebras. We relate this non-abelian tensor product with the universal central extension.

1.2 Preliminaries on Lie-Rinehart algebras

Most of the content of this section is well known, or follows from known results (see [4, 5, 11, 23]). We included it in order to fix terminology, notations and main examples. In what follows we fix a unital commutative ring K. All modules are considered over K. We write $\otimes$ and Hom instead of $\otimes_K$ and Hom_K.

1.2.1 Definitions, Examples

Let A be a unital commutative algebra over K. Then the set $\mathtt{Der}_K(\mathrm{A})$ of all K-derivations of A is a Lie K-algebra and an A-module simultaneously. These two structures are related by the following identity

$$[D, aD'] = a[D, D'] + D(a)D', \quad D, D' \in \mathtt{Der}_K(\mathrm{A}).$$

This leads to the notion below, which goes back to Herz under the name "pseudo-algèbre de Lie" and which is the algebraic counterpart of the Lie algebroid [20].

Definition 1.2.1. A *Lie-Rinehart* A-*algebra* consists of a Lie K-algebra L together with an A-module structure on L and a morphism, called the *anchor* map,

$$\alpha\colon L \to \mathtt{Der}_K(\mathrm{A}),$$

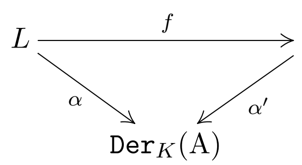
which is simultaneously a Lie algebra and A-module homomorphism such that

$$[x, ay] = a[x, y] + x(a)y.$$

Here $x, y \in L$, $a \in \mathrm{A}$ and we write $x(a)$ for $\alpha(x)(a)$ [11]. These objects are also known as (K, A)-Lie algebras [23] and d-Lie rings [22]. As stated in the literature ([10] for example), if L is a faithful A-module, the requirement of α being a Lie homomorphism follows from the other axioms.

Thus $\mathtt{Der}_K(\mathrm{A})$ with $\alpha = \mathsf{Id}_{\mathtt{Der}_K(\mathrm{A})}$ is a Lie-Rinehart A-algebra. Let us observe that Lie-Rinehart A-algebras with trivial homomorphism $\alpha\colon L \to \mathtt{Der}_K(\mathrm{A})$ are exactly Lie A-algebras. Therefore the concept of Lie-Rinehart algebras generalizes the concept of Lie A-algebras. If $\mathrm{A} = K$, then $\mathtt{Der}_K(\mathrm{A}){=}0$ and there is no difference between Lie and Lie-Rinehart algebras. If L is an A-module, then L is a trivial Lie-Rinehart A-algebra, that is L itself endowed with trivial Lie bracket and trivial anchor map.

If L and L' are Lie-Rinehart algebras, a *Lie-Rinehart* A-*algebra homomorphism* $f\colon L \to L'$ is a map, which is simultaneously a Lie K-algebra homomorphism and a homomorphism of A-modules. Furthermore it has to preserve the action on $\mathtt{Der}_K(\mathrm{A})$, in other words the diagram



commutes. We denote by $\mathtt{LR}_{\mathrm{A}K}$ the category of Lie-Rinehart A-algebras. We have the full inclusion

$$\mathtt{Lie}_{\mathrm{A}} \subset \mathtt{LR}_{\mathrm{A}K},$$

where $\mathtt{Lie}_{\mathrm{A}}$ denotes the category of Lie A-algebras.

It is important to see that the product in this category is not the cartesian product. For two Lie-Rinehart algebras L and M, the product in $\mathtt{LR}_{\mathrm{A}K}$ is $L \times_{\mathtt{Der}_K(\mathrm{A})} M = \{(l,m) \in L \times M \mid l(a) = m(a) \text{ for all } a \in \mathrm{A}\}$, where $L \times M$ denotes the cartesian product, with the action $(l,m)(a) = l(a) = m(a)$ for all $a \in \mathrm{A}$. Also note that the initial object is 0 but the terminal object is $\mathtt{Der}_K(\mathrm{A})$, then it does not have zero object (unless $K = \mathrm{A}$). This means that in general it is not a semi-abelian category in the sense of [15]. When we speak about a short exact sequence $I \to E \to L$ in $\mathtt{LR}_{\mathrm{A}K}$, we mean that the first homomorphism is injective and the second is surjective.

Let L be a Lie-Rinehart A-algebra. A Lie-Rinehart *subalgebra* M of L is a K-Lie subalgebra which is an A-module, with action induced by the inclusion in L. If M and N are two Lie-Rinehart subalgebras of L, we define the *commutator* of M and N, denoted by $\{M, N\}$ as the span as an A-module of the elements of the form $[x, y]$ where $x \in M$ and $y \in N$. Given a subalgebra M of L we say that it is a *quasi-ideal* if M is K-Lie ideal of L. Moreover, if the anchor map restricted to M is trivial, we will call it an *ideal*. In this way we have a correspondence between kernels of Lie-Rinehart homomorphisms (normal subobjects) and ideals. Another example is the centre of a Lie-Rinehart algebra, defined by

$$\mathsf{Z}_{\mathrm{A}}(L) = \{x \in L \mid [ax, z] = 0 \text{ and } x(a) = 0, \text{ for all } a \in \mathrm{A}, z \in L\}.$$

Note that L or $\{L, L\}$ are quasi-ideals of L but they are not necessarily ideals of L. We denote by L^{ab} the A-module $L/\{L, L\}$.

Example 1.2.2. The space of sections of a Lie algebroid is a Lie-Rinehart algebra (see [20]).

Example 1.2.3. If $\mathfrak{g}$ is a K-Lie algebra acting on a commutative K-algebra A by derivations (that is, a homomorphism of Lie K-algebras $\gamma\colon \mathfrak{g} \to \mathtt{Der}_K(\mathrm{A})$ is given), then *the transformation Lie-Rinehart algebra of* $(\mathfrak{g}, \mathrm{A})$ is $L = \mathrm{A} \otimes \mathfrak{g}$ with the Lie bracket

$$[a \otimes g, a' \otimes g'] := aa' \otimes [g, g'] + a\gamma(g)(a') \otimes g' - a'\gamma(g')(a) \otimes g,$$

where $a, a' \in \mathrm{A}$, $g, g' \in \mathfrak{g}$ and the action $\alpha\colon L \to \mathtt{Der}_K(\mathrm{A})$ is given by $\alpha(a \otimes g)(a') = a\gamma(g)(a')$.

Example 1.2.4. Let $\mathcal{M}$ be an A-module. The *Atiyah algebra* $\mathcal{A}_{\mathcal{M}}$ of $\mathcal{M}$ is the Lie-Rinehart A-algebra whose elements are pairs (f, D) with $f \in \mathrm{End}_K(\mathcal{M})$ and $D \in \mathtt{Der}_K(\mathrm{A})$ satisfying the following property:

$$f(am) = af(m) + D(a)m, \quad a \in \mathrm{A}, m \in \mathcal{M}\,.$$

$\mathcal{A}_{\mathcal{M}}$ is a Lie-Rinehart A-algebra with the Lie bracket

$$[(f, D), (f', D')] = ([f, f'], [D, D'])$$

and anchor map $(f, D) \mapsto D$ (see [16]).

Example 1.2.5. Let R be an associative algebra and $\mathsf{Z}(R)$ its centre (i.e. the elements $z \in R$ such that $zr = rz$ for all $r \in R$). Then $\mathrm{Der}_K(R)$ is a Lie-Rinehart algebra over $\mathsf{Z}(R)$ where the anchor $\alpha\colon \mathrm{Der}_K(R) \to \mathrm{Der}_K\big(\mathsf{Z}(R)\big)$ maps each derivation to its restriction in $\mathsf{Z}(R)$ (see [18]).

Example 1.2.6. Consider the K-algebra of dual numbers,

$$\mathrm{A} = K[\varepsilon] = K[X]/(X^2) = \big\{c_1 + c_2\varepsilon \mid c_1, c_2 \in K, \varepsilon^2 = 0\big\}.$$

We can endow to A with the Lie algebra structure given by the bracket:

$$[c_1 + c_2\varepsilon, c_1' + c_2'\varepsilon] = (c_1c_2' - c_2c_1')\varepsilon, \quad c_1 + c_2\varepsilon, c_1' + c_2'\varepsilon \in \mathrm{A}.$$

Thus A is a Lie-Rinehart A-algebra with anchor map $\alpha\colon \mathrm{A} \to \mathtt{Der}_K(\mathrm{A})$, $c_1 + c_2\varepsilon \mapsto \mathsf{ad}_{c_1}$, where $\mathsf{ad}_{c_1}(c_1' + c_2'\varepsilon) = [c_1, c_1' + c_2'\varepsilon]$ is the adjoint map of c_1.

Example 1.2.7. For a Lie-Rinehart algebra L, the A-module $L \oplus A$ with the bracket

$$[(x, a), (x', a')] = \big([x, x'], x(a') - x'(a)\big),$$

and anchor map $\widetilde{\alpha}\colon L \oplus \mathrm{A} \to \mathtt{Der}_K(\mathrm{A}), \widetilde{\alpha}(x, a) = \alpha_L(x)$ is a Lie-Rinehart algebra.

Example 1.2.8. Let us recall that a *Poisson algebra* is a commutative K-algebra P equipped with a Lie K-algebra structure such that the following identity holds

$$[a, bc] = b[a, c] + [a, b]c, \qquad a, b, c \in P.$$

There are (at least) three Lie-Rinehart algebras related to P. The first one is P itself considered as a P-module in an obvious way, where the action of P (as a Lie algebra) on P (as a commutative algebra) is given by the homomorphism $\mathsf{ad}\colon P \to \mathrm{Der}(P)$ given by $\mathsf{ad}(a) = [a, -] \in \mathrm{Der}(P)$. The second Lie-Rinehart algebra is the module of Kähler differentials Ω^1_P. It is easily shown (see [11]) that there is a unique Lie-Rinehart algebra structure on Ω^1_P such that $[da, db] = d[a, b]$ and such that the Lie algebra homomorphism $\Omega^1_P \to \mathrm{Der}(P)$ is given by $adb \mapsto a[b, -]$. To describe the third one, we need some preparations. We put

$$H^0_{\mathrm{Poiss}}(P, P) := \{a \in P \mid [a, -] = 0\}.$$

Then $H^0_{\mathrm{Poiss}}(P, P)$ contains the unit of P and is closed with respect to products, thus it is a subalgebra of P. A *Poisson derivation* of P is a linear map $D\colon P \to P$ which is a simultaneous derivation with respect to commutative and Lie algebra structures. We let $\mathsf{Der}_{\mathrm{Poiss}}(P)$ be the collection of all Poisson derivations of P. It is closed with respect to Lie bracket. Moreover if $a \in H^0_{\mathrm{Poiss}}(P, P)$ and $D \in \mathsf{Der}_{\mathrm{Poiss}}(P)$ then $aD \in \mathsf{Der}_{\mathrm{Poiss}}(P)$. It follows that $\mathsf{Der}_{\mathrm{Poiss}}(P)$ is a Lie-Rinehart $H^0_{\mathrm{Poiss}}(P, P)$- algebra. There is the following variant of the first construction in the graded case. Let $P_* = \bigoplus_{n \geqslant 0} P_n$ be a commutative graded K-algebra in the sense of commutative algebra (i.e. no signs are involved) and assume P_* is equipped with a Poisson algebra structure such that the bracket has degree (-1). Thus $[-, -]\colon P_n \otimes P_m \to P_{n+m-1}$. Then P_1 is a Lie-Rinehart P_0-algebra, where the Lie algebra homomorphism $P_1 \to \mathrm{Der}(P_0)$ is given by $a_1 \mapsto [a_1, -]$, $[a_1, -](a_0) = [a_1, a_0]$, where $a_i \in P_i$, $i = 0, 1$.

1.2.2 Actions and Semidirect Product of Lie-Rinehart algebras

Definition 1.2.9. Let $L \in \mathtt{LR}_{AK}$ and let R be a Lie A-algebra. We will say that L *acts* on R if it is given a K-linear map

$$L \otimes R \to R, \; (x, r) \mapsto x \circ r, \quad x \in L, r \in R$$

such that the following identities hold

(1) $x \circ (ar) = a(x \circ r) + x(a)r$,

(2) $[x, y] \circ r = x \circ (y \circ r) - y \circ (x \circ r)$,

(3) $x \circ [r, r'] = [x \circ r, r'] + [r, x \circ r']$,

(4) $ax \circ r = a(x \circ r)$,

where $a \in \mathrm{A}$, $x, y \in L$ and $r, r' \in R$.

Let us observe that (2) and (3) mean that L acts on R in the category of Lie K-algebras.

For a Lie-Rinehart algebra L and a Lie A-algebra R on which L acts we can form the *semidirect product* $L \rtimes R$ in the category of Lie K-algebras, which is $L \oplus R$ as a K-module, equipped with the following bracket

$$[(x, r), (y, r')] := ([x, y], [r, r'] + x \circ r' - y \circ r),$$

where $x, y \in L$ and $r, r' \in R$. With the A-module structure given by $a(x, r) = (ax, ar)$ and the anchor map $\widetilde{\alpha}(x, r) := \alpha(x)$ it has a Lie-Rinehart algebra structure, as seen in [4]. Observe that Example 1.2.7 is a particular case.

Definition 1.2.10 ([22]). *A left Lie-Rinehart* (A, L)*-module* over a Lie-Rinehart A-algebra L is a K-module $\mathcal{M}$ together with two operations

$$L \otimes \mathcal{M} \to \mathcal{M}, \qquad (x, m) \mapsto xm,$$

and

$$\mathrm{A} \otimes \mathcal{M} \to \mathcal{M}, \qquad (a, m) \mapsto am,$$

such that the first one makes $\mathcal{M}$ into a module over the Lie K-algebra L in the sense of the Lie algebra theory, while the second map makes $\mathcal{M}$ into an A-module and additionally the following compatibility conditions hold

$$\begin{aligned} (ax)(m) &= a(xm), \\ x(am) &= a(xm) + x(a)m, \qquad a \in \mathrm{A}, m \in \mathcal{M} \text{ and } x \in L. \end{aligned}$$

This definition can be seen as a particular case of Definition 1.2.9 where $\mathcal{M}$ is an abelian Lie A-algebra. Notice that a left Lie-Rinehart (A, L)-module is equivalent to give a morphism of Lie-Rinehart A-algebras $L \to \mathcal{A}_{\mathcal{M}}$ (see Example 1.2.4).

It is easy to see that A is a left Lie-Rinehart (A, L)-module for any Lie-Rinehart algebra L given by the anchor.

Definition 1.2.11 ([13]). *A right Lie-Rinehart* (A, L)*-module* over a Lie-Rinehart A-algebra L is a K-module $\mathcal{M}$ together with two operations

$$\mathcal{M} \otimes L \to \mathcal{M}, \qquad (m, x) \mapsto mx,$$

and

$$\mathrm{A} \otimes \mathcal{M} \to \mathcal{M}, \qquad (a, m) \mapsto am,$$

such that the first one makes $\mathcal{M}$ into a module over the Lie K-algebra L in the sense of the Lie algebra theory, while the second map makes $\mathcal{M}$ into an A-module and additionally the following compatibility conditions hold

$$(am)x = m(ax) = a(mx) - x(a)m, \qquad a \in \mathrm{A}, m \in \mathcal{M} \text{ and } x \in L.$$

Remark 1.2.12. The differences between the definitions of left and right (A, L)-module are significantly large. While in Lie algebras left and right L-modules are equivalent, in Lie-Rinehart that is not true. Concretely, A has a canonical left (A, L)-module structure but it does not hold a canonical right (A, L)-module structure. See [12] for a characterization of right (A, L)-module structures and see [19] for a concrete example.

1.2.3 Crossed Modules of Lie-Rinehart algebras

A *crossed module* $\partial \colon R \to L$ *of Lie-Rinehart* A*-algebras* (see [4]) consists of a Lie-Rinehart algebra L and a Lie A-algebra R together with the action of L on R and the Lie K-algebra homomorphism ∂ such that the following identities hold:

1. $\partial(x \circ r) = [x, \partial(r)]$,
2. $\partial(r') \circ r = [r', r]$,
3. $\partial(ar) = a\partial(r)$,
4. $\partial(r)(a) = 0$,

for all $a \in \mathrm{A}, r \in R$ and $x \in L$.

We can see some examples of crossed modules of Lie-Rinehart algebras.

1. For any Lie-Rinehart homomorphism $f \colon L \to R$, the diagram $\mathsf{Ker}\, f \to L$ is a crossed module of Lie-Rinehart algebras.

2. If M is an ideal of L, the inclusion $M \hookrightarrow L$ is a crossed module where the action of L on M is given by the Lie bracket.

3. If R is a left Lie-Rinehart (A, L)-module, the morphism $0\colon R \to L$ is a crossed module.

4. If $\partial\colon R \to L$ is a central epimorphism $\big($i.e. $\mathsf{Ker}\,\partial \subset \mathsf{Z}(R)\big)$ from a Lie A-algebra R to a Lie-Rinehart algebra L, ∂ is a crossed module where the action from L to R is given by $x \circ r = [r', r]$, such that $\partial(r') = x$.

1.2.4 Universal enveloping algebras and related constructions

There is a K-algebra $\mathsf{U}_{\mathrm{A}}\, L$ that has the property that the category of left (resp. right) $\mathsf{U}_{\mathrm{A}}\, L$-modules is equivalent to the category of left (resp. right) (A, L)-modules. Actually this algebra was constructed in [23]. We define the algebra $\mathsf{U}_{\mathrm{A}}\, L$ in terms of generators and relations. We have generators $i(x)$ for each $x \in L$ and $j(a)$ for each $a \in \mathrm{A}$. These generators must satisfy the following relations

$$\begin{aligned} j(1) &= 1, \qquad\qquad j(ab) = j(a)j(b), \\ i(ax) &= j(a)i(x), \\ i([x,y]) &= i(x)i(y) - i(y)i(x), \\ i(x)j(a) &= j(a)i(x) + j\big(x(a)\big). \end{aligned}$$

The first relations show that $j\colon \mathrm{A} \to \mathsf{U}_{\mathrm{A}}\, L$ is an algebra homomorphism.

Notice that in case of a trivial anchor one obtains the universal enveloping algebra of L as a Lie A-algebra.

We let V_n be the A-submodule spanned on all products $i(x_1)\cdots i(x_k)$, where $k \leqslant n$. Then

$$0 \subset \mathrm{A} = V_0 \subset V_1 \subset \cdots \subset V_n \subset \cdots \subset \mathsf{U}_{\mathrm{A}}\, L$$

defines an algebra filtration on $\mathsf{U}_{\mathrm{A}}\, L$. It is clear that $\mathsf{U}_{\mathrm{A}}\, L = \cup_{n \geqslant 0} V_n$. It follows from the third relation that the associated graded object $\mathsf{gr}_*(V)$ is a commutative A-algebra. In other words $\mathsf{U}_{\mathrm{A}}\, L$ is an almost commutative algebra in the following sense.

An *almost commutative algebra* is an associative K-algebra C together with a filtration

$$0 \subset \mathrm{A} = C_0 \subset C_1 \subset \cdots \subset C_n \subset \cdots \subset C = \bigcup_{n \geqslant 0} C_n$$

such that $C_n C_m \subset C_{n+m}$ and such that the associated graded object $\mathsf{gr}_*(C) = \bigoplus_{n\geqslant 0} C_n/C_{n-1}$ is a commutative A-algebra.

Remark 1.2.13. If C is an almost commutative algebra, then there is a well-defined bracket

$$[-,-]\colon \mathsf{gr}_n(C) \otimes \mathsf{gr}_m(C) \longrightarrow \mathsf{gr}_{n+m-1}(C)$$

which is given as follows. Let $a \in \mathsf{gr}_n(C)$ and $b \in \mathsf{gr}_m(C)$ and $\hat{a} \in C_n$ and $\hat{b} \in C_m$ be representatives of a and b. Since $\mathsf{gr}_*(C)$ is a commutative algebra it follows that $\hat{a}\hat{b} - \hat{b}\hat{a} \in C_{n+m-1}$ and the corresponding class in $\mathsf{gr}_{n+m-1}(C)$ is $[a, b]$. In this way we obtain a Poisson algebra structure on $\mathsf{gr}_*(C)$. Since the bracket is of degree (-1) it follows from Example 1.2.8 that $L = \mathsf{gr}_1(C)$ is a Lie-Rinehart $\mathrm{A} = \mathsf{gr}_0(C)$-algebra. Moreover the short exact sequence

$$\mathrm{A} \to C_1 \to L$$

is an abelian extension of Lie-Rinehart algebras (see below Definition 1.2.18).

Proposition 1.2.14. *The correspondence assigning C_1 to the almost commutative algebra C, defines a functor $LR\colon \mathtt{AComm}_{\mathrm{A}} \to \mathtt{LR}_{\mathrm{A}K}$.*

Proof. Let $f\colon C \to D$ be a morphism in $\mathtt{AComm}_{\mathrm{A}}$. Since f preserves the filtration, $f(C_1) \subseteq D_1$. Furthermore, $f(ax) = f(a)f(x) = af(x)$, for any $a \in C_0 = D_0$ and $x \in C_1$, and $f([x, y]) = f(xy - yx) = f(x)f(y) - f(y)f(x) = [f(x), f(y)]$, for $x, y \in C_1$. Hence the restriction of f to C_1, which we shall call $LR(f)$, is a morphism of K-Lie algebras and of A-modules such that the following diagram commutes in $\mathtt{Lie}_{\mathrm{K}}$,

$$\begin{array}{ccc} C_1 & \xrightarrow{LR(f)} & D_1 \\ & {\scriptstyle [\circ,-]}\searrow \quad \swarrow{\scriptstyle [\circ,-]} & \\ & \mathtt{Der}_K(\mathrm{A})\,. & \end{array}$$

Thus, $LR(f) \in \mathtt{LR}_{\mathrm{A}K}$.

On the other hand, it is clear that $LR(1_{C_1}) = 1_{C_1}$ and the following diagram commutes in K-mod,

$$\begin{array}{ccccc} C & \xrightarrow{f} & D & \xrightarrow{g} & E \\ {\scriptstyle i_C}\uparrow & & \uparrow{\scriptstyle i_D} & & \uparrow{\scriptstyle i_E} \\ C_1 & \xrightarrow{LR(f)} & D_1 & \xrightarrow{LR(g)} & E_1\,. \end{array}$$

Hence LR is functorial. □

Proposition 1.2.15. *The functor LR is right adjoint to the universal enveloping functor $\mathsf{U}_{\mathrm{A}}\colon \mathtt{LR}_{\mathrm{A}K} \to \mathtt{AComm}_{\mathrm{A}}$.*

Proof. Let $\Phi\colon \mathtt{AComm}_{\mathrm{A}}(\mathsf{U}_{\mathrm{A}}\,\mathrm{L},\mathrm{C}) \to \mathtt{LR}_{\mathrm{A}K}\big(\mathrm{L},\mathrm{LR}(\mathrm{C})\big)$ be the map given as follows. Since $\mathsf{U}_{\mathrm{A}}\,L$ is generated as a K-algebra by L and A, a morphism $f\colon \mathsf{U}_{\mathrm{A}}\,L \to C$ is completely determined by its restriction to L and A. Since $f(a) = a$ for every $a \in \mathrm{A}$, and $f(L) \subseteq f\big((\mathsf{U}_{\mathrm{A}}\,L)_1\big) \subseteq C_1$, it follows that the restriction of f to L, $\Phi f\colon L \to C_1 = LR(C)$ is a morphism of Lie-Rinehart algebras and Φ is a monomorphism.

Let $g\colon L \to C_1$ be a morphism in $\mathtt{LR}_{\mathrm{A}K}$. We build up $\widetilde{g}\colon \mathsf{U}_{\mathrm{A}}\,L \to C$ by $\widetilde{g}(ax_1\cdots x_m) := ag(x_1)\cdots g(x_m) \in C$. It is straightforward to see that $\widetilde{g}$ is a morphism in $\mathtt{AComm}_{\mathrm{A}}$ and $\Phi\widetilde{g} = g$. Hence Φ is bijective, and U_{A} and $\mathtt{LR}$ form an adjoint pair. □

There is another way to understand the universal enveloping algebra as an adjunction. We consider the category $\mathtt{Anc}_{\mathrm{A}}$ of anchored algebras, defined as A-algebras B equipped with an A-algebra morphism $\alpha\colon B \to \mathrm{End}(\mathrm{A})$, where the A-algebra structure on $\mathrm{End}(\mathrm{A})$ is given by $a \mapsto (l_a\colon a' \mapsto aa')$ and we construct a functor from $\mathtt{Anc}_{\mathrm{A}}$ to $\mathtt{LR}_{\mathrm{A}K}$ that sends an anchored algebra B to the A-submodule consisting of those elements $b \in B$ such that $\alpha(b) \in \mathtt{Der}_K(\mathrm{A})$. Then this functor is left adjoint to the universal enveloping functor (see [1]).

1.2.5 Free Lie-Rinehart Algebras

Here we follow [5]. Let ${}_{\mathrm{K}}\mathbf{mod}/\mathtt{Der}_{\mathrm{K}}(\mathrm{A})$ be the category of K-linear maps $\psi\colon V \to \mathtt{Der}_K(\mathrm{A})$, where V is a K-module. We have the functor

$$U\colon \mathtt{LR}_{\mathrm{A}K} \to {}_{\mathrm{K}}\mathbf{mod}/\mathtt{Der}_{\mathrm{K}}(\mathrm{A})$$

which assigns $\alpha\colon L \to \mathtt{Der}_K(\mathrm{A})$ to a Lie-Rinehart algebra L. A morphism $\psi \to \psi_1$ in ${}_{\mathrm{K}}\mathbf{mod}/\mathtt{Der}_{\mathrm{K}}(\mathrm{A})$ is a K-linear map $f\colon V \to V_1$ such that $\psi = \psi_1 f$. Now we construct the functor

$$F\colon {}_{\mathrm{K}}\mathbf{mod}/\mathtt{Der}_{\mathrm{K}}(\mathrm{A}) \to \mathtt{LR}_{\mathrm{A}K}$$

as follows. Let $\psi\colon V \to \mathtt{Der}_K(\mathrm{A})$ be a K-linear map. We let $\mathbf{L}(V)$ be the free Lie K-algebra generated by V. Then we have the unique Lie K-algebra homomorphism $\mathbf{L}(V) \to \mathtt{Der}_K(\mathrm{A})$ which extends the map ψ, which is still denoted by ψ. Now we can apply the construction from Example 1.2.3 to get a Lie-Rinehart algebra structure on $\mathrm{A} \otimes \mathbf{L}(V)$. We let $F(\psi)$ be this particular Lie-Rinehart algebra and we call it *the free Lie-Rinehart algebra generated by* ψ. In this way we obtain the functor F, which is the left adjoint to U.

Kapranov [17] defines a different concept of free Lie-Rinehart algebra as the adjoint of the forgetful functor $U' \colon \mathtt{LR}_{\mathrm{A}K} \to {}_{\mathrm{A}}\mathbf{mod}/\mathtt{Der}_{\mathrm{K}}(\mathrm{A})$. The relation between both constructions is given in [17, (2.2.8) Proposition].

1.2.6 Rinehart homology and cohomology of Lie-Rinehart algebras

Let $\mathcal{M}$ be a left Lie-Rinehart (A, L)-module. Let us recall the definition of the Rinehart cohomology $H^*_{\mathrm{Rin}}(L, \mathcal{M})$ of a Lie-Rinehart algebra L with coefficients in a left Lie-Rinehart module $\mathcal{M}$ (see [22, 23] and [5, 11]). We put

$$C^n_{\mathrm{A}}(L, \mathcal{M}) := \mathsf{Hom}_{\mathrm{A}}(\Lambda^n_{\mathrm{A}} L, \mathcal{M}), \qquad n \geqslant 0,$$

where $\Lambda^*_{\mathrm{A}}(V)$ denotes the exterior algebra over A generated by an A-module V. The coboundary map

$$\delta \colon C^{n-1}_{\mathrm{A}}(L, \mathcal{M}) \longrightarrow C^n_{\mathrm{A}}(L, \mathcal{M}),$$

is given by

$$(\delta f)(x_1, \dots, x_n) = \sum_{i=1}^{n} (-1)^{(i-1)}\, x_i\big(f(x_1, \dots, \hat{x_i}, \dots, x_n)\big) + \sum_{j<k} (-1)^{j+k} f([x_j, x_k], x_1, \dots, \hat{x_j}, \dots, \hat{x_k}, \dots, x_n),$$

where $x_1, \dots, x_n \in L, m \in \mathcal{M}, f \in C^{n-1}_{\mathrm{A}}(L, \mathcal{M})$.

We note that the differential δ is not A-linear unless L acts trivially on A.

For any left Lie-Rinehart (A, L)-module $\mathcal{M}$, the *Lie-Rinehart cohomology* is defined by

$$H^n_{\mathrm{Rin}}(L, \mathcal{M}) = H^n\big(C^n_{\mathrm{A}}(L, \mathcal{M})\big), \qquad n \geqslant 0.$$

Let $\mathcal{M}$ be a right Lie-Rinehart (A, L)-module. Let us recall the definition of the Rinehart homology $H_*^{\mathrm{Rin}}(L, \mathcal{M})$ of a Lie-Rinehart algebra L with coefficients in a right Lie-Rinehart module $\mathcal{M}$. We put

$$C_n^{\mathrm{A}}(L, \mathcal{M}) := \mathcal{M} \otimes_{\mathrm{A}} \Lambda^n_{\mathrm{A}} L, \qquad n \geqslant 0.$$

The boundary map

$$\partial\colon C_n^{\mathrm{A}}(L,\mathcal{M}) \longrightarrow C_{n-1}^{\mathrm{A}}(L,\mathcal{M}),$$

is given by

$$\begin{aligned}\partial\big(m\otimes_{\mathrm{A}}(x_1,\dots,x_n)\big) &= \sum_{i=1}^{n}(-1)^{(i-1)}\, mx_i\otimes_{\mathrm{A}}(x_1,\dots,\hat{x_i},\dots,x_n)\\ &\quad+\sum_{j<k}(-1)^{j+k}\, m\otimes_{\mathrm{A}}([x_j,x_k],x_1,\dots,\hat{x_j},\dots,\hat{x_k},\dots,x_n),\end{aligned}$$

where $x_1,\dots,x_n\in L, m\in\mathcal{M}$.

We note that the differential ∂ is not A-linear unless L acts trivially on A.

For any right Lie-Rinehart (A,L)-module $\mathcal{M}$, the *Lie-Rinehart homology* is defined by

$$H_n^{\mathrm{Rin}}(L,\mathcal{M}) = H_n\big(C_n^{\mathrm{A}}(L,\mathcal{M})\big),\qquad n\geqslant 0.$$

Let $\mathfrak{g}$ be a Lie algebra over K and let $\mathcal{M}$ be a $\mathfrak{g}$-module. Then we have the Chevalley-Eilenberg chain and cochain complexes $C_*^{\mathrm{Lie}}(\mathfrak{g},\mathcal{M})$ and $C^*_{\mathrm{Lie}}(\mathfrak{g},\mathcal{M})$, which compute the Lie algebra (co)homology (see [2]):

$$\begin{aligned}C_n^{\mathrm{Lie}}(\mathfrak{g},\mathcal{M}) &= \Lambda^n(\mathfrak{g})\otimes\mathcal{M},\\ C^n_{\mathrm{Lie}}(\mathfrak{g},\mathcal{M}) &= \mathsf{Hom}(\Lambda^n(\mathfrak{g}),\mathcal{M}).\end{aligned}$$

Here Λ^* denotes the exterior algebra defined over K.

One observes that if $\mathrm{A}=K$, then $H_*^{\mathrm{Rin}}(L,\mathcal{M})$ and $H^*_{\mathrm{Rin}}(L,\mathcal{M})$ generalize the classical definition of Lie algebra (co)homology.

For a general A by forgetting the A-module structure one obtains the canonical homomorphisms

$$H_*^{\mathrm{Lie}}(L,\mathcal{M})\to H_*^{\mathrm{Rin}}(L,\mathcal{M}),\qquad H^*_{\mathrm{Rin}}(L,\mathcal{M})\to H^*_{\mathrm{Lie}}(L,\mathcal{M}),$$

where $H_*^{\mathrm{Lie}}(L,\mathcal{M})$ and $H^*_{\mathrm{Lie}}(L,\mathcal{M})$ denote the homology and cohomology of L considered as a Lie K-algebra. On the other hand if A is a smooth commutative algebra, then $H^*_{\mathrm{Rin}}(\mathrm{Der}(\mathrm{A}),\mathrm{A})$ is isomorphic to the de Rham cohomology of A (see [23] and [11]).

Lemma 1.2.16. *Let* $\mathfrak{g}$ *be a Lie* K*-algebra acting on a commutative algebra* A *by derivations and let* L *be the transformation Lie-Rinehart algebra of* $(\mathfrak{g},\mathrm{A})$

(see Example 1.2.3). Then for any (right or left, depending on the situation) Lie-Rinehart (A, L)*-module* $\mathcal{M}$ *we have the canonical isomorphisms of complexes* $C^{\mathrm{A}}_*(L,\mathcal{M}) \cong C^{\mathrm{Lie}}_*(\mathfrak{g},\mathcal{M})$, $C^*_{\mathrm{A}}(L,\mathcal{M}) \cong C^n_{\mathrm{Lie}}(\mathfrak{g},\mathcal{M})$ *and in particular the isomorphisms*

$$H^{\mathrm{Rin}}_*(L,\mathcal{M}) \cong H^{\mathrm{Lie}}_*(\mathfrak{g},\mathcal{M}),$$
$$H^*_{\mathrm{Rin}}(L,\mathcal{M}) \cong H^*_{\mathrm{Lie}}(\mathfrak{g},\mathcal{M}).$$

Proof. Since $L = \mathrm{A}\otimes\mathfrak{g}$ we have $\Lambda^n_{\mathrm{A}} L \otimes_{\mathrm{A}} \mathcal{M} \cong \Lambda^n\mathfrak{g}\otimes\mathcal{M}$ and $\mathsf{Hom}_{\mathrm{A}}(\Lambda^n_{\mathrm{A}} L,\mathcal{M}) \cong \mathsf{Hom}(\Lambda^n\mathfrak{g},\mathcal{M})$ and lemma follows. □

Proposition 1.2.17. *Let* L *be a free Lie-Rinehart algebra generated by* $\psi\colon V \to \mathrm{Der}_K(\mathrm{A})$ *and let* $\mathcal{M}$ *be any (right or left, depending on the situation) Lie-Rinehart* (A, L)*-module. Then*

$$H^{\mathrm{Rin}}_n(L,\mathcal{M}) = 0, \qquad n > 1,$$
$$H^n_{\mathrm{Rin}}(L,\mathcal{M}) = 0, \qquad n > 1.$$

Proof. By our construction L is a transformation Lie-Rinehart algebra of $(\mathbf{L}(V), \mathrm{A})$. Thus we can apply Lemma 1.2.16 to get isomorphisms $H^{\mathrm{Rin}}_*(L,\mathcal{M}) \cong H^{\mathrm{Lie}}_*(\mathbf{L}(V),\mathcal{M})$ and $H^*_{\mathrm{Rin}}(L,\mathcal{M}) \cong H^*_{\mathrm{Lie}}(\mathbf{L}(V),\mathcal{M})$ and then we can use the well-known vanishing result for free Lie algebras (see [25]). □

1.2.7 Low degree homology groups of Lie-Rinehart algebras

Here we follow [5]. By definition, $H^{\mathrm{Rin}}_0(L,\mathcal{M}) = \frac{\mathcal{M}}{\mathcal{M}\circ L}$, is the module of *coinvariants* of $\mathcal{M}$, where $\mathcal{M}\circ L$ means the K-submodule of $\mathcal{M}$ generated by mx, $x \in L, m \in \mathcal{M}$, and $H^0_{\mathrm{Rin}}(L,\mathcal{M}) = \mathcal{M}^L = \{m \in \mathcal{M} \mid xm = 0 \text{ for all } x \in L\}$, is the *invariant* K*-submodule* of $\mathcal{M}$.

It follows from the definition that one has the following exact sequence

$$0 \to H^0_{\mathrm{Rin}}(L,\mathcal{M}) \to \mathcal{M} \xrightarrow{d} \mathrm{Der}_{\mathrm{A}}(L,\mathcal{M}) \to H^1_{\mathrm{Rin}}(L,\mathcal{M}) \to 0, \tag{1.2.1}$$

where $\mathrm{Der}_{\mathrm{A}}(L,\mathcal{M})$ consists of A-linear maps $d\colon L \to \mathcal{M}$ which are derivations from the Lie K-algebra L to $\mathcal{M}$. In other words d must satisfy the following conditions:

$$d(ax) = ad(x),$$
$$d([x,y]) = x\big(d(y)\big) - y\big(d(x)\big), \qquad a \in \mathrm{A},\ x, y \in L.$$

If $m \in \mathcal{M}$, the map $d_m\colon L \to \mathcal{M}, x \mapsto xm$, is a derivation. The maps d_m are called *inner derivations* of L into $\mathcal{M}$, and they form an K-submodule $\mathrm{IDer}_{\mathrm{A}}(L,\mathcal{M})$ of $\mathrm{Der}_{\mathrm{A}}(L,\mathcal{M})$. By (1.2.1), $H^1_{\mathrm{Rin}}(L,\mathcal{M}) \cong \mathrm{Der}_{\mathrm{A}}(L,\mathcal{M})/\,\mathrm{IDer}_{\mathrm{A}}(L,\mathcal{M})$.

If $\mathcal{M}$ is a trivial (A,L)-module, then $H^1_{\mathrm{Rin}}(L,\mathcal{M}) \cong \mathrm{Der}_{\mathrm{A}}(L,\mathcal{M}) \cong \mathsf{Hom}_{\mathrm{A}}(L^{\mathrm{ab}},\mathcal{M})$ and $H_1^{\mathrm{Rin}}(L,\mathcal{M}) \cong \frac{\mathcal{M}\otimes_{\mathrm{A}} L}{\mathcal{M}\otimes_{\mathrm{A}}\{L,L\}} \cong \mathcal{M}\otimes_{\mathrm{A}} L^{\mathrm{ab}}$.

1.2.8 Abelian extensions of Lie-Rinehart algebras

Definition 1.2.18. Let L be a Lie-Rinehart A-algebra and let $\mathcal{M}$ a left Lie-Rinehart (A,L)-module. An *abelian extension of L by $\mathcal{M}$* is a short exact sequence

$$\mathcal{M} \xrightarrow{i} L' \xrightarrow{\partial} L,$$

where L' is a Lie-Rinehart A-algebra and ∂ is a Lie-Rinehart algebra homomorphism. Moreover, i is an A-linear map and the following identities hold

$$\begin{aligned} [i(m),i(n)] &= 0, \\ [i(m),x'] &= \big(\partial(x')\big)(m), \qquad m,n \in \mathcal{M},\ x' \in L'. \end{aligned}$$

An abelian extension is called A-*split* if ∂ has an A-linear section.

Proposition 1.2.19 ([11, Theorem 2.6])**.** *If L is* A*-projective, then the cohomology $H^2_{\mathrm{Rin}}(L,\mathcal{M})$ classifies the abelian extensions*

$$\mathcal{M} \longrightarrow L' \longrightarrow L$$

of L by $\mathcal{M}$ in the category of Lie-Rinehart algebras that split in the category of A*-modules.*

The extension $\mathcal{M} \longrightarrow L \oplus \mathcal{M} \longrightarrow L$ represents $0 \in H^2_{\mathrm{Rin}}(L,\mathcal{M})$.

1.3 Universal central extensions of Lie-Rinehart algebras

1.3.1 Central extensions

An *extension* of a Lie-Rinehart algebra L is a short exact sequence

$$I \xrightarrow{i} E \xrightarrow{p} L, \tag{1.3.1}$$

where I, E and L are Lie-Rinehart algebras and i, p are Lie-Rinehart homomorphisms. Since $i\colon I \to i(I) = \mathsf{Ker}\, p$ is an isomorphism we shall identify I and $i(I)$. In other words, an extension of L is a surjective Lie-Rinehart homomorphism $p\colon E \to L$. If $p\colon E \to L$ and $p'\colon E' \to L$ are two extensions of L, a *homomorphism* from p to p' is a commutative diagram in $\mathtt{LR}_{\mathrm{A}K}$ of the form

$$\begin{array}{ccc} E & \xrightarrow{\;f\;} & E' \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p'} & \\ & L\,. & \end{array}$$

In particular,

$$\mathsf{Ker}\, f \subseteq f^{-1}(\mathsf{Ker}\, p') = \mathsf{Ker}\, p \quad \text{and} \quad E' = f(E) + \mathsf{Ker}\, p'. \tag{1.3.2}$$

An extension (1.3.1) is called *split* if there exists a Lie-Rinehart morphism $s\colon L \to E$, called *splitting homomorphism*, such that $ps = 1_L$. In this case, $E = I \oplus s(L)$ and $s\colon L \to s(L)$ is an isomorphism with inverse $p|_{s(L)}$. Moreover, $E \simeq I \rtimes L$, the semidirect product. In this way, semidirect products and split exact sequences are in a one to one correspondence. We point out that not every extension splits. We shall say that an extension *splits uniquely* whenever the splitting morphism is unique.

A *central extension of* L is an extension p such that $\mathsf{Ker}\, p \subseteq \mathsf{Z}_{\mathrm{A}}(E)$. In particular, if $p\colon\ E \underset{p}{\overset{s}{\leftrightarrows}} L$ is a split central extension, it is a product of K-Lie algebras $E = \mathsf{Ker}\, p \times L$, which is also a Lie-Rinehart algebra.

Proposition 1.3.1. *If L is* A*-projective, then $H^2_{\mathrm{Rin}}(L, I)$ classifies the central extensions*

$$I \longrightarrow E \longrightarrow L$$

of L by I.

Proof. Note that, if I is a trivial left Lie-Rinehart (A, L)-module, then an abelian extension of L by I is a central extension, and so the assertion follows by Proposition 1.2.19. □

In the A-projective case, the study of central extensions of Lie-Rinehart algebras can be seen as central extensions of Lie algebras, as we show in the next proposition.

Proposition 1.3.2. *Let $I = \mathrm{A} \cdot \alpha(L)(\mathrm{A})$ be the ideal generated by the action of the anchor map on* A*. Then $\overline{L} = L/IL$ is a Lie algebra over $\overline{\mathrm{A}} = \mathrm{A}/I$, and there is an equivalence of categories between on the one hand central extensions of the Lie-Rinehart algebra L that split as* A*-modules, and on the other hand central extensions of the $\overline{\mathrm{A}}$-Lie algebra $\overline{L} = L/IL$.*

Proof. If $\mathsf{Ker}\, p \longrightarrow E \overset{p}{\twoheadrightarrow} L$ is an A-split central extension, then centrality of $\mathsf{Ker}\, p$ implies $I \cdot \mathsf{Ker}\, p = \{0\}$. Now $I \cdot L \subset L$ is an ideal of K-Lie algebras, making $\overline{L} = L/IL$ a Lie-Rinehart algebra over $\overline{\mathrm{A}}$ with trivial anchor map, that is, an $\overline{\mathrm{A}}$-Lie algebra. Since p is A-split, we obtain a central extension $\mathsf{Ker}\, p \longrightarrow \overline{E} \overset{p}{\twoheadrightarrow} \overline{L}$ of $\overline{\mathrm{A}}$-Lie algebras. Conversely, every central extension $\mathsf{Ker}\, p \longrightarrow \tilde{E} \overset{p}{\twoheadrightarrow} \overline{L}$ of $\overline{\mathrm{A}}$-Lie algebras gives rise to the pullback central extension

$$\begin{array}{ccc} \hat{E} & \twoheadrightarrow & L \\ \downarrow & & \downarrow \\ \tilde{E} & \overset{p}{\twoheadrightarrow} & \overline{L}, \end{array}$$

where $\hat{E} = \{(x, y) \in \tilde{E} \times L \mid x \bmod \mathsf{Ker}\, p = y \bmod IL\}$. The correspondences $E \mapsto \overline{E}$ and $\tilde{E} \mapsto \hat{E}$ are functorial, and the natural transformations implementing the equivalence are the obvious ones. □

In the rest of this section we will study the general case of central extensions of Lie-Rinehart algebras.

A Lie-Rinehart A-algebra L is said *perfect* if $L = \{L, L\}$. A central extension E of L is called a *covering* if E is perfect; in that case, L is also perfect.

A central extension $\mathfrak{u}\colon \mathcal{L} \to L$ is called *universal* if there exists a unique homomorphism from $\mathfrak{u}$ to any other central extension of L. From the universal property of universal central extensions it immediately follows that two universal central extensions of L are isomorphic as extensions.

Lemma 1.3.3. ***(central trick)*** *Let $p\colon E \twoheadrightarrow L$ be a central extension.*

(a) If $p(x) = p(x')$ and $p(y) = p(y')$ then $[x, y] = [x', y']$ and for every $a \in \mathrm{A}$, $x(a) = x'(a)$.

(b) *If the following diagram commutes in* $\mathtt{LR}_{\mathrm{A}K}$,

$$P \overset{g}{\underset{f}{\rightrightarrows}} E \overset{p}{\twoheadrightarrow} L$$

then the restriction of both f *and* g *to* $\{P, P\}$ *agree; i.e.* $f|_{\{P,P\}} = g|_{\{P,P\}}$. *In particular, there exists at most one homomorphism from a covering* $C \longrightarrow\!\!\!\!\!\twoheadrightarrow L$ *to the central extension* $E \overset{p}{\twoheadrightarrow} L$.

Proof. (a) We have $x' = x + z$ and $y' = y + z'$ for some $z, z' \in \mathsf{Ker}\, p \subset \mathsf{Z}_{\mathrm{A}}(E)$, so it is clear that $[x', y'] = [x + z, y + z'] = [x, y]$. In addition, if p is a Lie-Rinehart homomorphism, the action on $\mathtt{Der}_K(\mathrm{A})$ must be preserved so $x(a) = x'(a)$.

(b) Using part (a), we have $g(a[x, y]) = a[g(x), g(y)] = a[f(x), f(y)] = f(a[x, y])$. □

Lemma 1.3.4. *Let* $p\colon E \twoheadrightarrow L$ *be a central extension where* L *is perfect. Then*

(a) $E = \{E, E\} + \mathsf{Ker}\, p$, *and* $p' = p|_{\{E,E\}} \colon \{E, E\} \twoheadrightarrow L$ *is a covering.*

(b) $\mathsf{Z}_{\mathrm{A}}(E) = p^{-1}\big(\mathsf{Z}_{\mathrm{A}}(L)\big)$ *and* $p\big(\mathsf{Z}_{\mathrm{A}}(E)\big) = \mathsf{Z}_{\mathrm{A}}(L)$.

(c) *If* $f\colon L \twoheadrightarrow M$ *is a central extension then so is* $fp\colon E \twoheadrightarrow M$.

(d) *If* $f\colon C \twoheadrightarrow L$ *is a covering and*

$$\begin{matrix} E & \xrightarrow{\quad g \quad} & C \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle f} & \\ & L & \end{matrix}$$

a morphism of extensions, then $g\colon E \twoheadrightarrow C$ *is a central extension. In particular,* g *is surjective.*

Proof. (a) Since $p(\{E, E\}) = \{L, L\} = L$ it follows easily that $E = \{E, E\} + \mathsf{Ker}\, p$ and $p|_{\{E,E\}}$ is clearly a covering.

(b) Let $z \in \mathsf{Z}_{\mathrm{A}}(E)$. For every $a \in \mathrm{A}$ we have $[az, E] = 0$, so $0 = [ap(z), p(E)] = [ap(z), L]$ then $p(z) \in \mathsf{Z}_{\mathrm{A}}(L)$. Conversely, let $z \in p^{-1}\big(\mathsf{Z}_{\mathrm{A}}(L)\big)$. For every $a \in \mathrm{A}$ we have $p([az, E]) = [ap(z), L] = 0$ so $[az, E] \subset \mathsf{Ker}\, p \subset$

$\mathsf{Z}_{\mathrm{A}}(E)$. Since $[az, E] = [az, \{E, E\} + \operatorname{Ker} p] = [az, \{E, E\}]$ we just have to check that $[az, \{E, E\}]$ is zero. Therefore, $\big[az, b[x, y]\big] = b\big[az, [x, y]\big] = b\big[x, [az, y]\big] + b\big[y, [x, az]\big] = 0$.

(c) The composition fp is clearly surjective and $\operatorname{Ker} fp = p^{-1}(\operatorname{Ker} f) \subset p^{-1}\big(\mathsf{Z}_{\mathrm{A}}(L)\big) = \mathsf{Z}_{\mathrm{A}}(E)$.

(d) Since $C = g(E) + \operatorname{Ker} f$, see (1.3.2), we have that $C = \{C, C\} = \{g(E), g(E)\} = g(\{E, E\})$ so g is surjective. Moreover, it is central since $\operatorname{Ker} g \subset \operatorname{Ker} p$. □

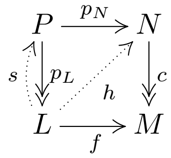
Corollary 1.3.5. *Let $L \in \mathtt{LR}_{\mathrm{A}K}$, arbitrary. If $L/\mathsf{Z}_{\mathrm{A}}(L)$ is perfect, then $\mathsf{Z}_{\mathrm{A}}(L/\mathsf{Z}_{\mathrm{A}}\, L) = 0$.*

Proof. It can be seen applying the second formula of Lemma 1.3.4(b) to the canonical map $p\colon\ L \twoheadrightarrow L/\mathsf{Z}_{\mathrm{A}}(L)$, which is a central extension. □

Lemma 1.3.6. ***(pullback Lemma)*** *Let $c\colon N \longrightarrow\!\!\!\!\!\rightarrow M$ be a central extension and $f\colon L \to M$ a morphism of Lie-Rinehart algebras, then,*

$$P := \{(l, n) \in L \times_{\mathtt{Der}_K(\mathrm{A})} N \mid f(l) = c(n)\}$$

is a Lie-Rinehart algebra and $p_L\colon P \to L$, $(l, n) \mapsto l$, is a central extension. This extension splits if and only if there exists a (unique) Lie-Rinehart morphism $h\colon L \to N$ such that $ch = f$.



Proof. It is clear that P is a Lie-Rinehart algebra with action $(l, n)(a) = l(a) = n(a)$, and p_L is a central extension. Moreover, a splitting homomorphism $s\colon L \to P$ exists (uniquely) if and only if there exists a (unique) Lie-Rinehart homomorphism $h\colon L \to N$ such that $s(l) = \big(l, h(l)\big)$ for all $l \in L$. □

Theorem 1.3.7. ***(characterization of universal central extensions)*** *For a Lie-Rinehart algebra L, there are equivalent:*

1. *Every central extension $L' \to L$ splits uniquely.*

2. $1_L\colon L \to L$ *is a universal central extension.*

Moreover, if $\mathfrak{u}\colon L \to M$ *is a central extension, then* (1) *and* (2) *are equivalent to*

3. $\mathfrak{u}\colon L \to M$ *is a universal central extension of* M*.*

In this case we also have,

(a) both L *and* M *are perfect and*

(b) $\mathsf{Z}_{\mathrm{A}}(L) = \mathfrak{u}^{-1}\big(\mathsf{Z}_{\mathrm{A}}(M)\big)$, $\mathfrak{u}\big(\mathsf{Z}_{\mathrm{A}}(L)\big) = \mathsf{Z}_{\mathrm{A}}(M)$*.*

Proof. By definition, (1) is equivalent to (2). Suppose that (3) holds, we want to prove (a). Let be the product as K-Lie algebras $L \times L/\{L, L\}$. In this case, this is actually a Lie-Rinehart algebra, with the usual operations and action $(x, y + \{L, L\})(a) = x(a)$, because

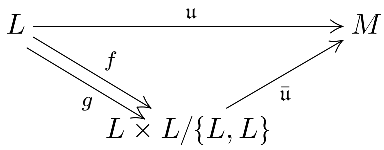
$$a[(x, y + \{L, L\}), (x', y' + \{L, L\})] = (a[x, x'], 0),$$

$$\begin{aligned}(x, y + \{L, L\})(a)(x', y' + \{L, L\}) &= (x(a)x', x(a)y' + \{L, L\}) \\ &= (x(a)x', [x, ay'] - a[x, y'] + \{L, L\}) = (x(a)x', 0),\end{aligned}$$

and therefore

$$[(x, y + \{L, L\}), a(x', y' + \{L, L\})] = (a[x, x'], 0) + (x(a)x', 0).$$

Now we can define the central extension $\bar{\mathfrak{u}}\colon L \times L/\{L, L\} \to M$, and two maps f and g



where $f(x) = (x, x + \{L, L\})$ and $g(x) = (x, 0)$. Since $\mathfrak{u}$ is universal, f and g must be equal, so $L/\{L, L\} = 0$. By the surjectivity of $\mathfrak{u}$, M is perfect too. The assertion (b) is consequence of Lemma 1.3.4(b).

We can prove now (3) $\Rightarrow$ (1). Let $f\colon L' \to L$ be a central extension. By Lemma 1.3.4(c) $\mathfrak{u}f$ is a central extension too, so by the universality of $\mathfrak{u}$, there exists $g\colon L \to L'$ such that $\mathfrak{u}fg = \mathfrak{u}$ and by Lemma 1.3.3(b) $fg = 1_L$.

To show (1) $\Rightarrow$ (3), for a central extension $f\colon N \to M$ we construct as in Lemma 1.3.6 the central extension p_L. Since p_L splits uniquely, by Lemma 1.3.6 there exists a unique map $h\colon L \to N$ such that $fh = \mathfrak{u}$. □

Corollary 1.3.8. *Let $f\colon E \to L$ and $g\colon L \to M$ be central extensions. Then $gf\colon E \to M$ is a universal central extension if and only if f is a universal central extension.*

Proof. The extension gf is central because E is perfect, so we can apply Lemma 1.3.4(c). Hence, f is universal if and only if $1_E\colon E \to E$ is universal, if and only if gf is universal. □

In the following corollary we assume that the perfect Lie-Rinehart algebras have universal central extensions, although in the next subsection we will prove that it is always the case.

Corollary 1.3.9. *Let L and L' be perfect Lie-Rinehart algebras, with universal central extensions $\mathfrak{u}\colon \mathcal{L} \to L$ and $\mathfrak{u}'\colon \mathcal{L}' \to L'$ respectively. Then*

$$L/\,\mathsf{Z}_{\mathrm{A}}(L) \cong L'/\,\mathsf{Z}_{\mathrm{A}}(L') \iff \mathcal{L} \cong \mathcal{L}'.$$

Proof. Given the diagram

$$\begin{array}{ccccc}
\mathcal{L} & \xrightarrow{\quad \mathfrak{u} \quad} & L & \xrightarrow{\quad \pi \quad} & L/\,\mathsf{Z}_{\mathrm{A}}(L) \\
{\scriptstyle \phi}\big\downarrow & & & & \big\downarrow{\scriptstyle \varphi} \\
\mathcal{L}' & \xrightarrow{\quad \mathfrak{u}' \quad} & L' & \xrightarrow{\quad \pi' \quad} & L'/\,\mathsf{Z}_{\mathrm{A}}(L')\,,
\end{array}$$

we assert that ϕ exists and is an isomorphism if and only if φ exists and is an isomorphism. Since $\pi\mathfrak{u}$ and $\pi'\mathfrak{u}'$ are universal central extensions by Corollary 1.3.8 and $L/\,\mathsf{Z}_{\mathrm{A}}(L)$ is isomorphic to $L'/\,\mathsf{Z}_{\mathrm{A}}(L')$, by the uniqueness of the universal central extension, $\mathcal{L} \cong \mathcal{L}'$. Conversely, by Corollary 1.3.5 $L/\,\mathsf{Z}_{\mathrm{A}}(L)$ is centreless. By Lemma 1.3.4(b) $\mathsf{Z}_{\mathrm{A}}(\mathcal{L}) = \mathsf{Ker}(\pi\mathfrak{u})$ and $\mathsf{Z}_{\mathrm{A}}(\mathcal{L}') = \mathsf{Ker}(\pi'\mathfrak{u}')$. Therefore, $\mathsf{Ker}(\pi'\mathfrak{u}'\phi) = \phi^{-1}\big(\mathsf{Ker}(\pi'\mathfrak{u}')\big) = \phi^{-1}\big(\mathsf{Z}_{\mathrm{A}}(\mathcal{L}')\big) = \mathsf{Z}_{\mathrm{A}}(\mathcal{L}) = \mathsf{Ker}(\pi\mathfrak{u})$. Since $\pi\mathfrak{u}$ and $\pi'\mathfrak{u}'\phi$ are surjective, φ exists and is an isomorphism. □

Note that the results obtained in this section generalize classic results of Lie algebras (see [24] and [25]).

1.3.2 The functor $\mathfrak{uce}_{\mathrm{A}}$

Let L be a Lie-Rinehart A-algebra. We denote by $\mathcal{S}_{\mathrm{A}}(L)$ the A-submodule of $\mathrm{A} \otimes_K L \otimes_K L$ spanned by the elements of the form

1. $a \otimes x \otimes x$
2. $a \otimes x \otimes y + a \otimes y \otimes x$
3. $a \otimes x \otimes [y, z] + a \otimes y \otimes [z, x] + a \otimes z \otimes [x, y]$
4. $a \otimes [x, y] \otimes [x', y'] + [x, y](a) \otimes x' \otimes y' - 1 \otimes [x, y] \otimes a[x', y']$

with $x, x', y, y', z \in L$ and $a \in \mathrm{A}$, and put

$$\mathfrak{uce}_{\mathrm{A}} L := \mathrm{A} \otimes_K L \otimes_K L / \mathcal{S}_{\mathrm{A}}(L).$$

We shall write $(a, x, y) := a \otimes x \otimes y + \mathcal{S}_{\mathrm{A}}(L) \in \mathfrak{uce}_{\mathrm{A}} L$.

By construction, the following identities hold in $\mathfrak{uce}_{\mathrm{A}}$:

1. $(a, x, y) = -(a, y, x)$,
2. $(a, x, [y, z]) + (a, y, [z, x]) + (a, z, [x, y]) = 0$,
3. $(1, [x, y], a[x', y']) = (a, [x, y], [x', y']) + ([x, y](a), x', y')$.

The map of A-modules $\mathrm{A} \otimes_K L \otimes_K L \to L$, determined by $(a, x, y) \mapsto a[x, y]$, vanishes on $\mathcal{S}_{\mathrm{A}}(L)$ and hence descends to a linear map

$$\mathfrak{u} \colon \mathfrak{uce}_{\mathrm{A}} L \to L.$$

Note that

$$\mathsf{Ker}\, \mathfrak{u} = \Big\{ \sum_i (a_i, x_i, y_i) \ \mid\ \sum_i a_i [x_i, y_i] = 0 \Big\}.$$

It is an easy but tedious calculation to see that the module $\mathfrak{uce}_{\mathrm{A}} L$ becomes a Lie-Rinehart algebra with the product

$$\begin{aligned} [(a, x, y), (a', x', y')] :=& (aa', [x, y], [x', y']) + (a[x, y](a'), x', y') \\ &- ([x', y'](a)a', x, y), \end{aligned}$$

and action

$$(a, x, y)(b) := a[x, y](b).$$

It then follows that $\mathfrak{u} \colon \mathfrak{uce}_{\mathrm{A}} L \to \{L, L\}$ is a central extension of $\{L, L\}$. In the case $\mathrm{A} = K$, we recover the well-known central extension $\mathfrak{uce}\, L \to [L, L]$ of Lie algebras, where $\mathfrak{uce}\, L$ denotes the universal central extension of L in the category of Lie algebras.

Let $f\colon L \to M$ be a Lie-Rinehart homomorphism. The A-module morphism $1_{\mathrm{A}} \otimes_K f \otimes_K f\colon \mathcal{S}_{\mathrm{A}}(L) \to \mathcal{S}_{\mathrm{A}}(M)$ induces an A-linear map

$$\mathfrak{uce}_{\mathrm{A}}(f)\colon \mathfrak{uce}_{\mathrm{A}}L \to \mathfrak{uce}_{\mathrm{A}}M, \quad (a, x, y) \mapsto \big(a, f(x), f(y)\big).$$

Note that the following diagram commutes by construction,

$$\begin{array}{ccc} \mathfrak{uce}_{\mathrm{A}}L & \xrightarrow{\mathfrak{uce}_{\mathrm{A}}(f)} & \mathfrak{uce}_{\mathrm{A}}M \\ {\scriptstyle \mathfrak{u}_L}\downarrow & & \downarrow{\scriptstyle \mathfrak{u}_M} \\ L & \xrightarrow[f]{} & M\,. \end{array} \tag{1.3.3}$$

To check that $\mathfrak{uce}_{\mathrm{A}}f$ is a morphism it suffices to show that

$$\mathfrak{uce}_{\mathrm{A}}(f)([(a, x, y), (a', x,' y,)]) = [\mathfrak{uce}_{\mathrm{A}}(f)(a, x, y), \mathfrak{uce}_{\mathrm{A}}(f)(a', x', y')],$$

which since f is a Lie-Rinehart homomorphism, we have that

$$a[x, y](a') = f(a[x, y])(a') = a[f(x), f(y)](a')$$

and the proof follows immediately.

Proposition 1.3.10. *Let $f\colon L \to M$ be a morphism of Lie-Rinehart algebras and suppose that $g\colon M' \to M$ is a central extension. Then there exists a homomorphism $\mathfrak{f}\colon \mathfrak{uce}_{\mathrm{A}}L \to M'$, making the following diagram commutative*

$$\begin{array}{ccc} \mathfrak{uce}_{\mathrm{A}}L & \xrightarrow{\mathfrak{f}} & M' \\ {\scriptstyle \mathfrak{u}}\downarrow & & \downarrow{\scriptstyle g} \\ L & \xrightarrow[f]{} & M. \end{array} \tag{1.3.4}$$

The map $\mathfrak{f}$ is uniquely determined on the commutator subalgebra $\{\mathfrak{uce}_{\mathrm{A}}L, \mathfrak{uce}_{\mathrm{A}}L\}$ by the commutativity of (1.3.4)*.*

Proof. Let $s\colon M \to M'$ a section of g in $\mathtt{Set}$. The map s may not be linear but we know that $s(km) - ks(m) \in \mathsf{Ker}\, g \subset \mathsf{Z}_{\mathrm{A}}(M')$ and $s(m+n) - s(m) - s(n) \in \mathsf{Ker}\, g \subset \mathsf{Z}_{\mathrm{A}}(M')$ for $k \in K$ and $m, n \in M$. Using this, we claim that the map

$$\begin{aligned} \mathrm{A} \times L \times L &\xrightarrow{\bar{f}} M' \\ (a, x, y) &\longmapsto a[sf(x), sf(y)], \end{aligned}$$

is bilinear, since

$$a[sf(kx), sf(y)] = a[sf(kx) - ksf(x) + ksf(x), sf(y)] = a[ksf(x), sf(y)].$$

The other property follows in the same way. By the universal property of tensor product, $\bar{f}$ defines a unique map between $\mathrm{A} \otimes_K L \otimes_K L$ and M'. In addition, the map is zero in $\mathcal{S}_{\mathrm{A}}(L)$, so it can be extended to $\mathfrak{f}\colon \mathfrak{uce}_{\mathrm{A}} L \to M'$, making the diagram commutative. It also preserves the anchor map because the section s must preserve it too. Using the property that $a[x,y](a') = f(a[x,y])(a') = a[f(x), f(y)](a')$, it follows immediately that $\mathfrak{f}$ is a Lie algebra homomorphism, hence it is a Lie-Rinehart algebra homomorphism that makes the diagram commutative. The uniqueness in $\{\mathfrak{uce}_{\mathrm{A}} L, \mathfrak{uce}_{\mathrm{A}} L\}$ follows from Lemma 1.3.3(b). □

Theorem 1.3.11. *Let L be a perfect Lie-Rinehart algebra. Then*

$$\mathsf{Ker}\,\mathfrak{u} \longrightarrow \mathfrak{uce}_{\mathrm{A}} L \xrightarrow{\mathfrak{u}} L,$$

is a universal central extension of L. Moreover, if L is centreless, then $\mathsf{Ker}\,\mathfrak{u} = \mathsf{Z}_{\mathrm{A}}(\mathfrak{uce}_{\mathrm{A}} L)$.

Proof. It can be seen that $\mathfrak{uce}_{\mathrm{A}}(\{L, L\}) \subset \{\mathfrak{uce}_{\mathrm{A}} L, \mathfrak{uce}_{\mathrm{A}} L\} \subset \mathfrak{uce}_{\mathrm{A}} L$. Thus when L is perfect, $\{\mathfrak{uce}_{\mathrm{A}} L, \mathfrak{uce}_{\mathrm{A}} L\} = \mathfrak{uce}_{\mathrm{A}} L$, so applying Proposition 1.3.10 for every central extension $f\colon M \to L$ we have a unique map $\mathfrak{f}\colon \mathfrak{uce}_{\mathrm{A}} L \to M$ making the diagram commutative. In other words, $\mathfrak{uce}_{\mathrm{A}} L$ is the universal central extension of L. □

Remark 1.3.12. In many algebraic structures as Lie algebras, $\mathsf{Ker}\,\mathfrak{u}$ is the second homology group with trivial coefficients. However, this is not possible here since we do not have a canonical right (A, L)-module structure in A as we have seen in Remark 1.2.12.

1.4 Lifting automorphisms and derivations

Let $f\colon L' \to L$ be a covering. Bringing back the commutative diagram (1.3.3) we get

$$\begin{array}{ccc} \mathcal{L}' & \xrightarrow{\ \mathfrak{f}\ } & \mathcal{L} \\ {\scriptstyle \mathfrak{u}'}\downarrow & & \downarrow{\scriptstyle \mathfrak{u}} \\ L' & \xrightarrow[\ f\ ]{} & L \end{array}$$

where $\mathcal{L}' = \mathfrak{uce}_{\mathrm{A}}(L')$, $\mathcal{L} = \mathfrak{uce}_{\mathrm{A}}(L)$ and $\mathfrak{u}' = \mathfrak{u}_{L'}$, $\mathfrak{u} = \mathfrak{u}_L$. Since both $\mathfrak{u}'$ and f are central extensions, by Corollary 1.3.8 we know that $f\mathfrak{u}' \colon \mathcal{L}' \to L$ is a universal central extension of L. By the uniqueness of the universal central extension, we know that $\mathcal{L}' \cong \mathcal{L}$. In addition, since $\mathfrak{f}$ is a morphism from the universal central extension $f\mathfrak{u}'$ to the central extension $\mathfrak{u}$, it must be an isomorphism. Therefore, we get a covering $\mathfrak{u}'\mathfrak{f}^{-1} \colon \mathcal{L} \to L'$ with kernel

$$C := \mathsf{Ker}(\mathfrak{u}'\mathfrak{f}^{-1}) = \mathfrak{f}(\mathsf{Ker}\,\mathfrak{u}').$$

1.4.1 Lifting of automorphisms

Theorem 1.4.1. *Let $f \colon L' \to L$ be a covering.*

(a) Let $h \in \mathsf{Aut}(L)$. Then there exists $h' \in \mathsf{Aut}(L')$ such that the diagram

$$\begin{CD} L' @>{f}>> L \\ @V{h'}VV @VV{h}V \\ L' @>>{f}> L \end{CD} \tag{1.4.1}$$

commutes if and only if $\mathfrak{uce}_{\mathrm{A}}(h)(C) = C$. Moreover, h' is uniquely determined by the diagram (1.4.1) *and $h'(\mathsf{Ker}\,f) = \mathsf{Ker}\,f$.*

(b) The group homomorphism

$$\{h \in \mathsf{Aut}(L) \mid \mathfrak{uce}_{\mathrm{A}}(h)(C) = C\} \to \{g \in \mathsf{Aut}(L') \mid g(\mathsf{Ker}\,f) = \mathsf{Ker}\,f\}, \quad h \mapsto h',$$

is a group isomorphism.

Proof. (a) If h' exists, it is a morphism from the covering hf to the covering f so by Lemma 1.3.3(b) it is uniquely determined by the commutativity of the diagram (1.4.1). Let us suppose that h' exists. If we apply the $\mathfrak{uce}_{\mathrm{A}}$ functor to the diagram (1.4.1), we obtain the commutative diagram

$$\begin{CD} \mathcal{L}' @>{\mathfrak{f}}>> \mathcal{L} \\ @V{\mathfrak{uce}_{\mathrm{A}}(h')}VV @VV{\mathfrak{uce}_{\mathrm{A}}(h)}V \\ \mathcal{L}' @>>{\mathfrak{f}}> \mathcal{L}\,. \end{CD}$$

In this way,

$$\begin{aligned}\mathfrak{uce}_{\mathrm{A}}(h)(C) &= \mathfrak{uce}_{\mathrm{A}}(h)\big(\mathfrak{f}(\mathsf{Ker}\,\mathfrak{u}')\big) = (\mathfrak{uce}_{\mathrm{A}}(h)\,\mathfrak{f})(\mathsf{Ker}\,\mathfrak{u}') \\ &= (\mathfrak{f}\,\mathfrak{uce}_{\mathrm{A}}(h'))(\mathsf{Ker}\,\mathfrak{u}') = \mathfrak{f}(\mathsf{Ker}\,\mathfrak{u}') = C.\end{aligned}$$

Suppose now that $\mathfrak{uce}_{\mathrm{A}}(h)(C) = C$. We obtain the commutative diagram

$$\begin{array}{ccccc} \mathcal{L} & \xrightarrow{\mathfrak{u}'\mathfrak{f}^{-1}} & L' & \xrightarrow{f} & L \\ {\scriptstyle \mathfrak{uce}_{\mathrm{A}}(h)}\downarrow & & \vdots & & \downarrow{\scriptstyle h} \\ \mathcal{L} & \xrightarrow{\mathfrak{u}'\mathfrak{f}^{-1}} & L' & \xrightarrow{f} & L\,. \end{array} \tag{1.4.2}$$

If $\mathfrak{uce}_{\mathrm{A}}(h)(C) = C$, the kernel of the epimorphism $\mathfrak{u}'\mathfrak{f}^{-1} \circ \mathfrak{uce}_{\mathrm{A}}(h)$ is C, i.e. the kernel of $\mathfrak{u}'\mathfrak{f}^{-1}$. In this way, we obtain an automorphism $h' \colon L' \to L'$ such that (1.4.2) commutes. The condition that $h'(\mathsf{Ker}\,f) = \mathsf{Ker}\,f$ follows immediately by the commutativity of (1.4.1).

(b) The map is well defined and injective by part (a) of the theorem. Let $g \in \mathsf{Aut}(L')$ such that $g(\mathsf{Ker}\,f) = \mathsf{Ker}\,f$. It descends to $h \in \mathsf{Aut}(L)$ such that $fg = hf$. Again by (a), g must be the lifting of h and since the lifting exists it follows that $\mathfrak{uce}_{\mathrm{A}}(h)(C) = C$. □

Corollary 1.4.2. *If L is perfect, the map*

$$\mathsf{Aut}(L) \to \{g \in \mathsf{Aut}\big(\mathfrak{uce}_{\mathrm{A}}(L)\big) \mid g(\mathsf{Ker}\,\mathfrak{u}) = \mathsf{Ker}\,\mathfrak{u}\}, \quad f \mapsto \mathfrak{uce}_{\mathrm{A}}(f),$$

is a group isomorphism. Moreover, if L is centreless, then $\mathsf{Aut}(L) \cong \mathsf{Aut}\big(\mathfrak{uce}_{\mathrm{A}}(L)\big)$.

Proof. Applying the last theorem to the covering $\mathfrak{u} \colon \mathfrak{uce}_{\mathrm{A}}(L) \to L$, we have that $\mathfrak{u}'$ is the identity map, so $C = 0$ and the corollary follows immediately. By Lemma 1.3.4(b), if L is perfect we have that $\mathsf{Ker}\,\mathfrak{u} = \mathsf{Z}_{\mathrm{A}}\left(\mathfrak{uce}_{\mathrm{A}}(L)\right)$ and since every automorphism leaves the centre invariant it is straightforward that $\mathsf{Aut}(L) \cong \mathsf{Aut}\big(\mathfrak{uce}_{\mathrm{A}}(L)\big)$. □

1.4.2 Lifting of derivations

Definition 1.4.3. Let L be a Lie-Rinehart algebra over A. A *derivation of L* is a pair $D := (\delta, \delta_0)$, where $\delta \colon L \to L$ is a derivation of L as a K-Lie algebra,

$\delta_0 \in \mathtt{Der}_K(\mathrm{A})$ and the following identities hold:

$$\delta(ax) = a\delta(x) + \delta_0(a)x,$$
$$\delta_0\big(x(a)\big) = x\big(\delta_0(a)\big) + \delta(x)(a),$$

with $a \in \mathrm{A}$ and $x \in L$. Note that the second identity means that the following diagram commutes,

$$\begin{array}{ccc} L & \xrightarrow{\ \delta\ } & L \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha} \\ \mathtt{Der}_K(\mathrm{A}) & \xrightarrow[{[\delta_0,-]}]{} & \mathtt{Der}_K(\mathrm{A})\,. \end{array}$$

For any $x \in L$ we have an associated derivation (δ, δ_0) where $\delta(y) = [x, y]$ and $\delta_0(a) = x(a)$.

We shall write $\mathfrak{Der}(L)$ the A-module of all derivations of the Lie-Rinehart algebra L. Observe that $\mathfrak{Der}(L)$, with Lie bracket $[(\delta, \delta_0), (\delta', \delta'_0)] = ([\delta, \delta'], [\delta_0, \delta'_0])$ and anchor map $\mathfrak{Der}(L) \to \mathtt{Der}_K(\mathrm{A})$, $(\delta, \delta_0) \mapsto \delta_0$ is a Lie-Rinehart algebra over A. In the particular case of $K = \mathrm{A}$, we recover the notion of Lie derivation.

Recall that if $(\delta, \delta_0) \in \mathfrak{Der}(L)$ we have that $\delta\big(\mathsf{Z}_{\mathrm{A}}(L)\big) \subset \mathsf{Z}_{\mathrm{A}}(L)$, since if $x \in \mathsf{Z}_{\mathrm{A}}(L)$,

$$[a\delta(x), z] = \delta([ax, z]) - [ax, \delta(z)] - [\delta_0(a)x, z] = 0.$$

Lemma 1.4.4. *Let $f\colon L \to M$ be a central extension of Lie-Rinehart algebras. If (δ, δ_0) and (δ', δ_0) are derivations of L such that $f\delta = f\delta'$ then $\delta|_{\{L,L\}} = \delta'|_{\{L,L\}}$.*

Proof. Since f is a central extension, $a[\delta(x), y] = a[\delta'(x), y]$ and $a[x, \delta(y)] = a[x, \delta'(y)]$. Thus,

$$\begin{aligned}\delta(a[x,y]) &= \delta_0(a)[x,y] + a[\delta(x),y] + a[x,\delta(y)] \\ &= \delta_0(a)[x,y] + a[\delta'(x),y] + a[x,\delta'(y)] = \delta'(a[x,y]),\end{aligned}$$

for all $a \in A$ and $x, y \in L$. □

Given a derivation $D = (\delta, \delta_0) \in \mathfrak{Der}(L)$, one can define $\mathfrak{uce}_{\mathrm{A}}(D) = (\delta^{\mathfrak{u}}, \delta_0)$, where $\delta^{\mathfrak{u}}$ is defined on generators as $(a, x, y) \mapsto (\delta_0(a), x, y) + (a, \delta(x), y) +$

$\big(a, x, \delta(y)\big)$. It is a straightforward verification that the map $\mathfrak{uce}_{\mathrm{A}}(D)$ is also a derivation of the Lie-Rinehart algebra $\mathfrak{uce}_{\mathrm{A}}(L)$ and yields the following commutative diagram

$$\begin{array}{ccc} \mathfrak{uce}_{\mathrm{A}}(L) & \xrightarrow{\ \delta^{\mathfrak{u}}\ } & \mathfrak{uce}_{\mathrm{A}}(L) \\ {\scriptstyle \mathfrak{u}}\downarrow & & \downarrow{\scriptstyle \mathfrak{u}} \\ L & \xrightarrow[\ \delta\ ]{} & L \end{array}$$

leaving $\mathsf{Ker}\,\mathfrak{u}$ invariant. Moreover, the map

$$\mathfrak{uce}_{\mathrm{A}} \colon \mathfrak{Der}(L) \to \{(\gamma, \gamma_0) \in \mathfrak{Der}\big(\mathfrak{uce}_{\mathrm{A}}(L)\big) \mid \gamma(\mathsf{Ker}\,\mathfrak{u}) \subset (\mathsf{Ker}\,\mathfrak{u})\}, \;\; D \mapsto \mathfrak{uce}_{\mathrm{A}}(D),$$

is a Lie-Rinehart homomorphism, and its kernel is contained in the subalgebra of those derivations vanishing on $\{L, L\}$.

In the following lemma we check how $\mathfrak{uce}_{\mathrm{A}}$ operates with derivations.

Lemma 1.4.5. *Let $f \colon L \to M$ be a morphism of perfect Lie-Rinehart algebras and let $(\delta_L, \delta_0) \in \mathfrak{Der}(L)$ and $(\delta_M, \delta_0) \in \mathfrak{Der}(M)$ be such that $f\delta_L = \delta_M f$. Then, we have that*

$$\mathfrak{uce}_{\mathrm{A}}(f)\delta_L^{\mathfrak{u}} = \delta_M^{\mathfrak{u}}\mathfrak{uce}_{\mathrm{A}}(f).$$

Proof. It suffices to check it for an element $(a, x, y) \in \mathfrak{uce}_{\mathrm{A}}(L)$.

$$\begin{aligned} \mathfrak{uce}_{\mathrm{A}}(f)\delta_L^{\mathfrak{u}}(a, x, y) &= \mathfrak{uce}_{\mathrm{A}}(f)\big((\delta_0(a), x, y) + (a, \delta_L(x), y) + (a, x, \delta_L(y))\big) \\ &= \big(\delta_0(a), f(x), f(y)\big) + \big(a, f\big(\delta_L(x)\big), f(y)\big) + \big(a, f(x), f\big(\delta_L(y)\big)\big) \\ &= \big(\delta_0(a), f(x), f(y)\big) + \big(a, \delta_M\big(f(x)\big), f(y)\big) + \big(a, f(x), \delta_M\big(f(y)\big)\big) \\ &= \delta_M\big(a, f(x), f(y)\big) \\ &= \delta_M\mathfrak{uce}_{\mathrm{A}}(f)(a, x, y). \end{aligned}$$

□

We will state now the analogue of Theorem 1.4.1 for derivations.

Theorem 1.4.6. *Let $f \colon L' \to L$ be a covering of the Lie-Rinehart algebra L and as before, we denote $C = \mathfrak{uce}_{\mathrm{A}}(f)(\mathsf{Ker}\,\mathfrak{u}')$.*

(a) A derivation $D = (\delta, \delta_0) \in \mathfrak{Der}(L)$ lifts to a derivation $D' = (\delta', \delta_0)$ of L' satisfying $\delta f = f\delta'$ if and only if the derivation $\delta^{\mathfrak{u}}(C) \subset C$. Moreover, δ' is uniquely determined and leaves $\mathsf{Ker}\, f$ invariant.

(b) The map

$$\{(\delta,\delta_0) \in \mathfrak{Der}(L) \mid \mathfrak{uce}_{\mathrm{A}}(\delta^{\mathfrak{u}})(C) \subset C\} \longrightarrow \{(\eta,\eta_0) \in \mathfrak{Der}(L') \mid \eta^{\mathfrak{u}}(\mathsf{Ker}\, f) \subset \mathsf{Ker}\, f\}$$
$$(\delta,\delta_0) \longmapsto (\delta',\delta_0)$$

is an isomorphism of Lie-Rinehart algebras.

(c) In particular, from the covering $\mathfrak{u}\colon \mathfrak{uce}_{\mathrm{A}}(L) \to L$ *we obtain that the map*

$$\mathfrak{uce}_{\mathrm{A}}\colon \mathfrak{Der}(L) \to \{(\gamma,\gamma_0) \in \mathfrak{Der}\big(\mathfrak{uce}_{\mathrm{A}}(L)\big) \mid \gamma(\mathsf{Ker}\,\mathfrak{u}) \subset \mathsf{Ker}\,\mathfrak{u}\}$$

is an isomorphism. Moreover, if L *is centreless, we have that* $\mathfrak{Der}(L) \cong \mathfrak{Der}\big(\mathfrak{uce}_{\mathrm{A}}(L)\big)$.

Proof. (a) If the derivation $D' = (\delta',\delta_0)$ exists, by Lemma 1.4.4 it is unique. Using Lemma 1.4.5 we have that $\delta^{\mathfrak{u}}(C) = \mathfrak{uce}_{\mathrm{A}}(\delta^{\mathfrak{u}})\big(\mathfrak{f}(\mathsf{Ker}\,\mathfrak{u}')\big) = (\mathfrak{uce}_{\mathrm{A}}(\delta^{\mathfrak{u}})\,\mathfrak{f})(\mathsf{Ker}\,\mathfrak{u}') = (\mathfrak{f} \circ \mathfrak{uce}_{\mathrm{A}}(\delta'^{\mathfrak{u}}))(\mathsf{Ker}\,\mathfrak{u}') \subset \mathfrak{f}(\mathsf{Ker}\,\mathfrak{u}') = C$. Conversely, if $\delta^{\mathfrak{u}}(C) \subset C$, taking the analogue for derivations of diagram (1.4.1), it follows immediately.

(b) The map is well defined and injective by part (a) and surjective by Lemma 1.4.4.

(c) We have that $\mathfrak{u}'$ is the identity map, so $C = 0$. In the case that $\mathsf{Z}_{\mathrm{A}}(L) = 0$, we have that $\mathsf{Ker}\,\mathfrak{u} = \mathsf{Z}_{\mathrm{A}}\big(\mathfrak{uce}_{\mathrm{A}}(L)\big)$, then $\mathfrak{Der}(L) \cong \mathfrak{Der}\big(\mathfrak{uce}_{\mathrm{A}}(L)\big)$ follows immediately, since every derivation leaves the centre invariant. □

1.4.3 Universal central extensions of split exact sequences

Theorem 1.4.7. *Let* $L \xrightarrow{\ f\ } M \underset{g}{\overset{s}{\rightleftarrows}} N$ *be a split short exact sequence of perfect Lie-Rinehart algebras. We have the following commutative diagram*

$$\begin{array}{ccccc}
\mathfrak{uce}_{\mathrm{A}}(L) & \xrightarrow{\ \varphi\ } & \mathfrak{uce}_{\mathrm{A}}(M) & \underset{\gamma}{\overset{\sigma}{\rightleftarrows}} & \mathfrak{uce}_{\mathrm{A}}(N) \\
\downarrow{\scriptstyle \mathfrak{u}_L} & & \downarrow{\scriptstyle \mathfrak{u}_M} & & \downarrow{\scriptstyle \mathfrak{u}_N} \\
L & \xrightarrow{\ f\ } & M & \underset{g}{\overset{s}{\rightleftarrows}} & N
\end{array}$$

where $\mathfrak{uce}_{\mathrm{A}}(M)$ *is a semidirect product*

$$\mathfrak{uce}_{\mathrm{A}}(M) = \varphi\big(\mathfrak{uce}_{\mathrm{A}}(L)\big) \rtimes \sigma\big(\mathfrak{uce}_{\mathrm{A}}(N)\big),$$

and

$$\mathsf{Ker}\,\mathfrak{u}_M = \varphi(\mathsf{Ker}\,\mathfrak{u}_L) \oplus \sigma(\mathsf{Ker}\,\mathfrak{u}_N).$$

We know that $M \cong L \rtimes N$ since the bottom row exact sequence splits. If moreover $M = L \oplus N$, i.e. $[f(L), s(N)] = \{0\}$, we have

$$\mathfrak{uce}_{\mathrm{A}}(L \oplus N) \cong \mathfrak{uce}_{\mathrm{A}}(L) \oplus \mathfrak{uce}_{\mathrm{A}}(N).$$

Proof. In order to simplify the notation, we can interpret f and s as identifications, so we will write l for $f(l)$ and n for $s(n)$. Given an element of the form $(a, \tilde{n}, \tilde{l}) \in \mathfrak{uce}_{\mathrm{A}}(M)$, with $\tilde{n} \in N$ and $\tilde{l} \in L$, by the perfectness of L and the properties of $\mathfrak{uce}_{\mathrm{A}}(M)$, we have

$$\begin{aligned}(a, \tilde{n}, \tilde{l}) &= (a, b[n, n'], c[l, l']) \\ &= (ac, b[n, n'], [l, l']) + (ab[n, n'](c), l, l') \\ &= (ac, [b[n, n'], l], l') + (ac, l, [b[n, n'], l]) + (ab[n, n'](c), l, l'),\end{aligned}$$

which means that $(A, N, L) \subset (A, L, L)$, so $\mathfrak{uce}_{\mathrm{A}}(M) = (A, L, L) + (A, N, N)$. By definition, $(A, L, L) = \varphi\big(\mathfrak{uce}_{\mathrm{A}}(L)\big)$ and $(A, N, N) = \sigma\big(\mathfrak{uce}_{\mathrm{A}}(N)\big)$. Now since $\gamma\sigma$ is the identity map, we know that $\mathfrak{uce}_{\mathrm{A}}(M) \cong \mathsf{Ker}\,\gamma \rtimes \sigma\big(\mathfrak{uce}_{\mathrm{A}}(N)\big)$. In this way, $\sigma\big(\mathfrak{uce}_{\mathrm{A}}(N)\big) \cong \mathfrak{uce}_{\mathrm{A}}(N)$ and since $(A, L, L) \subset \mathsf{Ker}\,\gamma$ it follows that $\mathsf{Ker}\,\gamma = (A, L, L) = \varphi\big(\mathfrak{uce}_{\mathrm{A}}(L)\big)$, so we have that $\mathfrak{uce}_{\mathrm{A}}(M) = \varphi\big(\mathfrak{uce}_{\mathrm{A}}(L)\big) \rtimes \sigma\big(\mathfrak{uce}_{\mathrm{A}}(N)\big)$.

Every element of $\mathfrak{uce}_{\mathrm{A}}(M)$ has the form $\varphi(l) + \sigma(n)$ where $l \in \mathfrak{uce}_{\mathrm{A}}(L)$ and $n \in \mathfrak{uce}_{\mathrm{A}}(N)$. This means that any element of $\mathfrak{uce}_{\mathrm{A}}(M)$ is in $\mathsf{Ker}\,\mathfrak{u}_M$ if and only if $0 = \mathfrak{u}_M\varphi(l) = \mathfrak{u}_L(l)$ and $0 = \mathfrak{u}_M\sigma(n) = \mathfrak{u}_N(n)$, so $\mathsf{Ker}\,\mathfrak{u}_M = \varphi(\mathsf{Ker}\,\mathfrak{u}_L) \oplus \sigma(\mathsf{Ker}\,\mathfrak{u}_N)$.

In the particular case that $M = L \oplus N$, we can define the induced map

$$\varphi \oplus \sigma \colon \mathfrak{uce}_{\mathrm{A}}(L) \oplus \mathfrak{uce}_{\mathrm{A}}(N) \to \mathfrak{uce}_{\mathrm{A}}(M),$$

and it is an easy computation that it is a Lie-Rinehart algebra morphism. Moreover, $\mathsf{Ker}(\varphi \oplus \sigma) = \mathsf{Ker}\,\varphi$. Given $l \in \mathsf{Ker}\,\varphi$, $\mathfrak{u}_M\varphi(l) = 0 = \mathfrak{u}_L(l)$ so $l \in \mathsf{Ker}\,\mathfrak{u}_L \in \mathsf{Z}_{\mathrm{A}}(L)$, which means that $\varphi \oplus \sigma$ is a covering. We can use now Theorem 1.3.7(2) to see that $\varphi \oplus \sigma$ is an isomorphism completing the proof. □

1.5 Non-abelian tensor product of Lie-Rinehart algebras

A non-abelian tensor product of Lie algebras was introduced by Ellis [6]. Here we adapt some of his results to the case of Lie-Rinehart algebras, in order to use them to obtain a description of universal central extensions in this category.

Definition 1.5.1. Let $L, M \in \mathtt{LR}_{\mathrm{A}K}$. By a *quasi-action of L on M*, we mean a K-bilinear map, $L \times M \to M$, $(x, m) \mapsto {}^{x}m$, satisfying

1. ${}^{x}(am) = a({}^{x}m) + x(a)m$,
2. ${}^{[x,y]}m = {}^{x}({}^{y}m) - {}^{y}({}^{x}m)$,
3. ${}^{x}[m, n] = [{}^{x}m, n] + [m, {}^{x}n]$,

for all $a \in \mathrm{A}$, $x, y \in L$ and $m, n \in M$. We will say that L *quasi-acts* on M. For example, if L is a subalgebra of some Lie-Rinehart algebra $\mathcal{L}$ and M is an ideal of $\mathcal{L}$ then the bracket in $\mathcal{L}$ yields a quasi-action of L on M. In the particular case of $K = \mathrm{A}$, a quasi-action is the same as a Lie action.

Remark 1.5.2. The category of Lie-Rinehart algebras does not fit into the theory of semi-abelian categories in the sense of [15] so the notion of internal action [14] cannot be recovered. We could be tempted to add another identity such as ${}^{ax}m = a({}^{x}m)$ (identity (4) in Definition 1.2.9), but then a Lie-Rinehart algebra would not act on itself via the bracket. On the other hand, since the normal subobjects are very limited (they have to be Lie A-algebras), to form a semidirect product compatible with the notion of split extensions we recover the notion of action of Definition 1.2.9, but this is not useful to our proposes since we cannot form an action over an arbitrary Lie-Rinehart algebra.

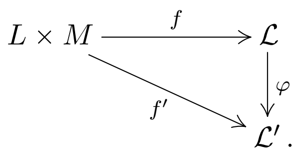
If we have a quasi-action of L on M and a quasi-action of M on L, for any Lie-Rinehart algebra $\mathcal{L}$ we call a K-bilinear function $f\colon L \times M \to \mathcal{L}$ a *Lie-Rinehart pairing* if

1. $\alpha_{\mathcal{L}}\big(f(x, m)\big) = [\alpha_L(x), \alpha_M(m)]$,
2. $f([x, y], m) = f(x, {}^{y}m) - f(y, {}^{x}m)$,
3. $f(x, [m, n]) = f({}^{n}x, m) - f({}^{m}x, n)$,

4. $f\big(a({}^m x), b({}^y n)\big) = -ab[f(x,m), f(y,n)] - a[\alpha_L(x), \alpha_M(m)](b)f(y,n)$
$+ [\alpha_L(y), \alpha_M(n)](a)bf(x,m)$,

for all $a, b \in \mathrm{A}$, $x, y \in L$, $m, n \in M$, and where α_L, α_M denote the anchor maps corresponding to L and M, respectively. If M and N are quasi-ideals of a Lie-Rinehart algebra L, then $f\colon M \times N \to M \cap N, (m,n) \mapsto [m,n]$ is a Lie-Rinehart pairing.

We say that a Lie-Rinehart pairing $f\colon L \times M \to \mathcal{L}$ is *universal* if for any other Lie-Rinehart pairing $f'\colon L \times M \to \mathcal{L}'$ there is a unique Lie-Rinehart homomorphism $\varphi\colon \mathcal{L} \to \mathcal{L}'$ making commutative the diagram:



The Lie-Rinehart algebra $\mathcal{L}$ is unique up to isomorphism which we will describe as the non-abelian tensor product of L and M.

Definition 1.5.3. Let L and M be a pair of Lie-Rinehart algebras together with a quasi-action of L on M and a quasi-action of M on L. We define the *non-abelian tensor product of L and M in $\mathtt{LR}_{\mathrm{A}K}$, $L \otimes M$*, as the Lie-Rinehart A-algebra spanned as an A-module by the symbols $x \otimes m$, and subject only to the relations:

1. $ka(x \otimes m) = a(kx \otimes m) = a(x \otimes km)$,

2. $x \otimes (m + n) = x \otimes m + x \otimes n$,
$(x + y) \otimes m = x \otimes m + y \otimes m$,

3. $[x, y] \otimes m = x \otimes {}^y m - y \otimes {}^x m$,
$x \otimes [m, n] = {}^n x \otimes m - {}^m x \otimes n$,

4. $a({}^m x) \otimes b({}^y n) = ab({}^m x \otimes {}^y n) - a[\alpha_L(x), \alpha_M(m)](b)(y \otimes n)$
$+ b[\alpha_L(y), \alpha_M(n)](a)(x \otimes m)$,

for every $k \in K$, $a, b \in \mathrm{A}$, $x, y \in L$ and $m, n \in M$, with the induced bracket $[a(x \otimes m), b(y \otimes n)] = -a({}^m x) \otimes b({}^y n)$ and anchor map $\alpha\colon L \otimes M \to \mathtt{Der}_K(\mathrm{A})$ given by $\alpha\big(a(x \otimes m)\big) := a[\alpha_L(x), \alpha_M(m)]$.

This way, the map $f\colon L \times M \to L \otimes M$ which sends (x, m) to $x \otimes m$ is a universal Lie-Rinehart pairing by construction.

Definition 1.5.4. Two quasi-actions $L \times M \to M$ and $M \times L \to L$ are said to be *compatible* if for all $x, y \in L$ and $m, n \in M$,

1. $-\alpha_L({}^{m}x) = \alpha_M({}^{x}m) = [\alpha_L(x), \alpha_M(m)]$,

2. ${}^{({}^{m}x)}n = [n, {}^{x}m]$,

3. ${}^{({}^{x}m)}y = [y, {}^{m}x]$.

This is the case, for example, if L and M are both quasi-ideals of some Lie-Rinehart algebra and the quasi-actions are given by multiplication. We can see another example of compatible quasi-actions when $\partial\colon L \to N$ and $\partial'\colon M \to N$ are crossed modules. In this case, L and M quasi-act on each other via the action of N. These quasi-actions are compatible. If $\mathrm{A} = K$ then we recover the notion of compatible actions between Lie algebras [6].

From this point on we shall assume that all quasi-actions are compatible.

Proposition 1.5.5. *Let $\mu\colon L \otimes M \to L$ and $\nu\colon L \otimes M \to M$ be the homomorphisms defined on generators by $\mu\big(a(x \otimes m)\big) = -a({}^{m}x)$ and $\nu\big(a(x \otimes m)\big) = a({}^{x}m)$. They are Lie-Rinehart homomorphisms and the following diagram is commutative:*

$$\begin{array}{ccc} L \otimes M & \xrightarrow{\ \nu\ } & M \\ {\scriptstyle\mu}\downarrow & \searrow^{\alpha} & \downarrow{\scriptstyle\alpha_M} \\ L & \xrightarrow[\ \alpha_L\ ]{} & \mathtt{Der}_K(\mathrm{A})\,. \end{array}$$

We can relate the Lie-Rinehart tensor product $L \otimes M$ with the tensor product of L and M as an A-module. We will denote it by $L \underset{\mathrm{mod}}{\otimes} M$ the A-module generated by the symbols $x \otimes m$ subject to the relations

1. $k(x \otimes m) = kx \otimes m = x \otimes km$,

2. $x \otimes (m + n) = x \otimes m + x \otimes n$,
   $(x + y) \otimes m = x \otimes m + y \otimes m$,

for every $k \in K$, $x, y \in L$ and $m, n \in M$.

Proposition 1.5.6. *The canonical map* $L \underset{\mathrm{mod}}{\otimes} M \to L \otimes M$ *is a surjective* A*-module homomorphism. In addition, if* L *and* M *quasi-act trivially on each other, there is an isomorphism of* A*-modules:*

$$L \otimes M \cong \frac{L}{[L,L]} \underset{\mathrm{mod}}{\otimes} \frac{M}{[M,M]},$$

where $L/[L,L]$ *and* $M/[M,M]$ *denote the abelianization as* K*-Lie algebras.*

Proof. The identity (3) of Definition 1.5.3 is satisfied since the quasi-action is trivial and the identity (4) is consequence of this fact and of Definition 1.5.4 (1). □

Proposition 1.5.7. *The Lie-Rinehart algebras* $L \otimes M$ *and* $M \otimes L$ *are isomorphic.*

Proof. The map $f\colon L \times M \to M \otimes L$ which sends $(x,m) \to m \otimes x$ is a Lie-Rinehart pairing, then by the universal property of $L \otimes M$ there is a Lie-Rinehart homomorphism $L \otimes M \to M \otimes L$. In a similar way, we can construct the inverse $M \otimes L \to L \otimes M$ and establish an isomorphism. □

Proposition 1.5.8. *Consider the following short exact sequence of Lie-Rinehart algebras*

$$L \xrightarrow{f} M \xrightarrow{g} N,$$

and assume that there are compatible quasi-actions of a Lie-Rinehart algebra P *on* L*,* M *and* N*, and of* L, M, N *on* P*. Suppose also that the Lie-Rinehart morphisms* f *and* g *preserve these quasi-actions, i.e.* $f({}^{p}x) = {}^{p}f(x)$, ${}^{x}p = {}^{f(x)}p$ *and* $g({}^{p}m) = {}^{p}g(m)$, ${}^{m}p = {}^{g(m)}p$, *where* $x \in L$, $m \in M$ *and* $p \in P$*. Then, the following sequence is exact*

$$L \otimes P \xrightarrow{f \otimes 1} M \otimes P \xrightarrow{g \otimes 1} N \otimes P, \quad \textit{with } g \otimes 1 \textit{ surjective.}$$

Proof. Since f and g preserve the quasi-actions, it is easy to see that $f \otimes 1$ and $g \otimes 1$ are Lie-Rinehart algebra morphisms. Furthermore, the morphism $g \otimes 1$ is clearly surjective, and $\mathsf{Im}(f \otimes 1) \subset \mathsf{Ker}(g \otimes 1)$. Since $gf = 0$, we have that $f(x)(a) = 0$ for every $a \in \mathrm{A}$ and $x \in L$. This means that $(f \otimes 1)(x \otimes p)(a) = [\alpha_M(f(x)), \alpha_P(p)](a) = 0$. Moreover, $\mathsf{Im}(f \otimes 1)$ is an A-module and preserves the Lie bracket since f and g preserve the quasi-actions, so $\mathsf{Im}(f \otimes 1)$ is an ideal. Then to prove the other inclusion, we will show that $M \otimes P/\mathsf{Im}(f \otimes 1) \cong N \otimes P$. Since $\mathsf{Im}(f \otimes 1) \subset \mathsf{Ker}(g \otimes 1)$ we have a natural epimorphism $\phi\colon M \otimes P/\mathsf{Im}(f \otimes 1) \to N \otimes P$. Now we define the map $\psi\colon N \times P \to M \otimes P/\mathsf{Im}(f \otimes 1)$ such that $\varphi(n, p) = m \otimes p + \mathsf{Im}(f \otimes 1)$ where m is such that $g(m) = n$. It follows that it is a well-defined Lie-Rinehart pairing, so by the universality of the tensor product, there exists a unique Lie-Rinehart morphism $\varphi\colon N \otimes P \to M \otimes P/\mathsf{Im}(f \otimes 1)$, and it is straightforward that ϕ and φ are inverse morphisms. □

Proposition 1.5.9. *Given a perfect Lie-Rinehart algebra L, the non-abelian tensor product $L \otimes L$ is the universal central extension of L, where the quasi-action of L on itself is the Lie bracket.*

Proof. It is routine to check that $L \otimes L \to L$ is a central extension. To see the universality, given a central extension $p\colon M \to L$, we pick a section in `Set`, $s\colon L \to M$. We define now a map $f\colon L \times L \to M$ by $f(x, y) = [s(x), s(y)]$. Doing the same trick as in Proposition 1.3.10, we see that is a Lie-Rinehart pairing, so it can be extended to $L \otimes L \to M$. Since L is perfect, we saw in Lemma 1.3.3 that this map is unique. □

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Chapter 2

Non-abelian tensor product and homology of Lie superalgebras

Abstract

We introduce the non-abelian tensor product of Lie superalgebras and study some of its properties. We use it to describe the universal central extensions of Lie superalgebras. We present the low-dimensional non-abelian homology of Lie superalgebras and establish its relationship with the cyclic homology of associative superalgebras. We also define the non-abelian exterior product and give an analogue of Miller's theorem, Hopf formula and a six-term exact sequence for the homology of Lie superalgebras.

Reference

X. García-Martínez, E. Khmaladze, and M. Ladra, *Non-abelian tensor product and homology of Lie superalgebras*, J. Algebra **440** (2015), 464–488.

2.1 Introduction

In [1], Brown and Loday introduced the non-abelian tensor product of groups in the context of an application in homotopy theory. Analogous theories of non-abelian tensor product have been developed in other algebraic structures such as Lie algebras [8] and Lie–Rinehart algebras [4]. In [8], Ellis investigated the main properties of the non-abelian tensor product of Lie algebras and its relation to the low-dimensional homology of Lie algebras. In particular, he described the universal central extension of a perfect Lie algebra via the non-abelian tensor product. In [7], the non-abelian exterior product of Lie algebras is introduced and a six-term exact sequence relating low-dimensional homologies is obtained. In [10], using the non-abelian tensor product, Guin defined the non-abelian low-dimensional homology of Lie algebras and compared these groups with the cyclic homology and Milnor additive K-theory of associative algebras.

The theory of Lie superalgebras, also called $\mathbb{Z}_2$-graded Lie algebras, has aroused much interest both in mathematics and physics. Lie superalgebras play a very important role in theoretical physics since they are used to describe supersymmetry in a mathematical framework. A comprehensive description of the mathematical theory of Lie superalgebras is given in [14], containing the complete classification of all finite-dimensional simple Lie superalgebras over an algebraically closed field of characteristic zero. In the last few years, the theory of Lie superalgebras has experienced a remarkable evolution obtaining many results on representation theory and classification, most of them extending well-known facts on Lie algebras.

In this paper we develop the non-abelian tensor product and the low-dimensional non-abelian homology of Lie superalgebras, generalizing the corresponding notions for Lie algebras, with applications in universal central extensions and homology of Lie superalgebras and cyclic homology of associative superalgebras.

The organization of this paper is as follows: after this introduction, in Section 2.2 we give some definitions and necessary well-known results for the development of the paper. We also introduce actions and crossed modules of Lie superalgebras. In Section 2.3 we introduce the non-abelian tensor product of Lie superalgebras, we establish its principal properties such as right exactness and relation with the tensor product of supermodules. We describe the universal central extension of a perfect Lie superalgebra via the non-abelian tensor product (Theorem 2.4.1). In particular, applying this theorem, we obtain that $\mathfrak{st}(m,n,A)$ is the universal central extension of $\mathfrak{sl}(m,n,A)$, for $m+n \geqslant 5$, where A is a unital associative superalgebra. We also study nilpotency and solvability of the non-abelian tensor product of Lie superalgebras (Theorem 2.3.9). Using the non-abelian tensor product, in Section 2.5 we introduce the low-dimensional non-abelian homology of Lie superalgebras with coefficients in crossed modules. We show that, if the crossed module is a supermodule, then the non-abelian homology is the usual homology of Lie superalgebras. Then we apply this non-abelian homology to relate cyclic homology and Milnor cyclic homology of associative superalgebras, extending the results of [10]. Finally, in the last section we construct the non-abelian exterior product of Lie superalgebras and we use it to obtain Miller's type theorem for free Lie superalgebras, Hopf formula and a six-term exact sequence in the homology of Lie superalgebras.

Conventions and notations

Throughout this paper we denote by $\mathbb{K}$ a unital commutative ring unless otherwise stated. All modules and algebras are defined over $\mathbb{K}$. We write $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ and use its standard field structure. We put $(-1)^{\bar{0}} = 1$ and $(-1)^{\bar{1}} = -1$.

By *a supermodule* M we mean a module endowed with a $\mathbb{Z}_2$-gradation: $M = M_{\bar{0}} \oplus M_{\bar{1}}$. We call elements of $M_{\bar{0}}$ (resp. $M_{\bar{1}}$) even (resp. odd). Non-zero elements of $M_{\bar{0}} \cup M_{\bar{1}}$ will be called *homogeneous*. For a homogeneous $m \in M_{\bar{\alpha}}$, $\bar{\alpha} \in \mathbb{Z}_2$, its degree will be denoted by $|m|$. We adopt the convention that whenever the degree function occurs in a formula, the corresponding elements are supposed to be homogeneous. By *a homomorphism of supermodules* $f \colon M \to N$ of degree $|f| \in \mathbb{Z}_2$ we mean a linear map satisfying $f(M_{\bar{\alpha}}) \subseteq N_{\bar{\alpha}+|f|}$. In particular, if $|f| = \bar{0}$, then the homomorphism f will be called *of even grade (or even linear map)*.

By *a superalgebra* A we mean a supermodule $A = A_{\bar{0}} \oplus A_{\bar{1}}$ equipped with a bilinear multiplication satisfying $A_{\bar{\alpha}} A_{\bar{\beta}} \subseteq A_{\bar{\alpha}+\bar{\beta}}$, for $\bar{\alpha}, \bar{\beta} \in \mathbb{Z}_2$.

2.2 Preliminaries on Lie Superalgebras

In this section we review some terminology on Lie superalgebras and recall notions used in the paper. We mainly follow [2, 18], although with some modifications. We also introduce notions of actions and crossed modules of Lie superalgebras.

2.2.1 Definition and some examples of Lie superalgebras

Definition 2.2.1. A *Lie superalgebra* is a superalgebra $M = M_{\bar{0}} \oplus M_{\bar{1}}$ with a multiplication denoted by $[\ ,\ ]$, called bracket operation, satisfying the following identities:

$$\begin{aligned}
[x, y] &= -(-1)^{|x||y|}[y, x], \\
\big[x, [y, z]\big] &= \big[[x, y], z\big] + (-1)^{|x||y|}\big[y, [x, z]\big], \\
[m_{\bar{0}}, m_{\bar{0}}] &= 0,
\end{aligned}$$

for all homogeneous elements $x, y, z \in M$ and $m_{\bar{0}} \in M_{\bar{0}}$.

Note that the last equation is an immediate consequence of the first one in the case 2 has an inverse in $\mathbb{K}$. Moreover, it can be easily seen that the second equation is equivalent to the graded Jacobi identity

$$(-1)^{|x||z|}\big[x, [y, z]\big] + (-1)^{|y||x|}\big[y, [z, x]\big] + (-1)^{|z||y|}\big[z, [x, y]\big] = 0.$$

For a Lie superalgebra $M = M_{\bar{0}} \oplus M_{\bar{1}}$, the even part $M_{\bar{0}}$ is a Lie algebra. Hence, if $M_{\bar{1}} = 0$, then M is just a Lie algebra. A Lie superalgebra M without even part, i. e., $M_{\bar{0}} = 0$, is an *abelian Lie superalgebra*, that is, $[x, y] = 0$ for all $x, y \in M$.

A *Lie superalgebra homomorphism* $f \colon M \to M'$ is a supermodule homomorphism of even grade such that $f[x, y] = [f(x), f(y)]$ for all $x, y \in M$.

Example 2.2.2.

(i) Any associative superalgebra A can be considered as a Lie superalgebra with the bracket

$$[a, b] = ab - (-1)^{|a||b|}ba.$$

(ii) Let m, n be positive integers and A a unital associative superalgebra. Consider the algebra $\mathcal{M}(m, n, A)$ of all $(m+n) \times (m+n)$-matrices with entries in A and with the usual product of matrices. A $\mathbb{Z}_2$-gradation is defined as follows: homogeneous elements are matrices $E_{ij}(a)$ having the homogeneous element $a \in A$ at the position (i, j) and zero elsewhere, and $|E_{ij}(a)| = |i| + |j| + |a|$, where $|i| = \bar{0}$ if $1 \leqslant i \leqslant m$ and $|i| = \bar{1}$ if $m + 1 \leqslant i \leqslant m + n$. With this gradation, $\mathcal{M}(m, n, A)$ turns out to be an associative superalgebra. The corresponding Lie superalgebra will be denoted by $\mathfrak{gl}(m, n, A)$.

(iii) Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a supermodule. Then the supermodule $\operatorname{End}_{\mathbb{K}}(V)$ of all linear endomorphisms $V \to V$ (of both degrees 0 and 1) has a structure of an associative superalgebra with respect to composition (see [2]) and hence becomes a Lie superalgebra. In particular, if the ground ring $\mathbb{K}$ is a field, and m, n are dimensions of $V_{\bar{0}}$ and $V_{\bar{1}}$ respectively, then choosing a homogeneous basis of V ordered such that even elements stand before odd, the elements of $\mathrm{End}_{\mathbb{K}}(V)$ can be seen as $(m+n)\times(m+n)$-square matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where a, b, c and d are respectively $m \times m$, $m \times n$, $n \times m$ and $n \times n$ matrices with entries in $\mathbb{K}$. The even elements are the matrices with $b = c = 0$ and the odd elements are matrices with $a = d = 0$.

Let M and N be two submodules of a Lie superalgebra P. We denote by $[M, N]$ the submodule of P spanned by all elements $[m, n]$ with $m \in M$ and $n \in N$. A $\mathbb{Z}_2$-graded submodule M is a *graded ideal* of P if $[M, P] \subseteq M$. In particular, the submodule $\mathsf{Z}(P) = \{c \in P : [c, p] = 0 \text{ for all } p \in P\}$ is a graded ideal and it is called the *centre* of P. Clearly if M and N are graded ideals of P, then so is $[M, N]$.

Let M be a Lie superalgebra and $D \in \mathrm{End}_{\mathbb{K}}(M)$. We say that D is a *derivation* if for all $x, y \in M$

$$D([x,y]) = [D(x), y] + (-1)^{|D||x|}[x, D(y)].$$

We denote by $\big(\mathrm{Der}_{\mathbb{K}}(M)\big)_{\bar{\alpha}}$ the set of homogeneous derivations of degree $\bar{\alpha} \in \mathbb{Z}_2$. One verifies that the supermodule of derivations

$$\mathrm{Der}_{\mathbb{K}}(M) = \big(\mathrm{Der}_{\mathbb{K}}(M)\big)_{\bar{0}} \oplus \big(\mathrm{Der}_{\mathbb{K}}(M)\big)_{\bar{1}}$$

is a subalgebra of the Lie superalgebra $\mathrm{End}_{\mathbb{K}}(M)$.

2.2.2 Actions and crossed modules of Lie superalgebras

Definition 2.2.3. Let P and M be two Lie superalgebras. By an *action* of P on M we mean a $\mathbb{K}$-bilinear map of even grade,

$$P \times M \to M, \quad (p, m) \mapsto {}^{p}m,$$

such that

(i) ${}^{[p,p']}m = {}^{p}({}^{p'}m) - (-1)^{|p||p'|}\, {}^{p'}({}^{p}m),$

(ii) ${}^{p}[m,m'] = [{}^{p}m,m'] + (-1)^{|p||m|}[m,{}^{p}m']$,

for all homogeneous $p,p' \in P$ and $m,m' \in M$.

The action is called *trivial* if ${}^{p}m = 0$ for all $p \in P$ and $m \in M$.

For example, if M is a graded ideal and P is a subalgebra of a Lie superalgebra Q, then the bracket in Q induces an action of P on M.

Note that the action of P on M is the same as a Lie superalgebra homomorphism $P \to \mathrm{Der}_{\mathbb{K}}(M)$.

Remark 2.2.4. If M is an abelian Lie superalgebra enriched with an action of a Lie superalgebra P, then M has a structure of a supermodule over P (P-supermodule, for short) (see e. g. [18]), that is, there is a $\mathbb{K}$-bilinear map of even grade $P \times M \to M$, $(p,m) \mapsto pm$, such that

$$[p,p']m = p(p'm) - (-1)^{|p||p'|}p'(pm),$$

for all homogeneous $p,p' \in P$ and $m \in M$.

Note that a P-supermodule M is the same as a $\mathbb{K}$-supermodule M together with a Lie superalgebra homomorphism $P \to \mathrm{End}_{\mathbb{K}}(M)$.

Definition 2.2.5. Given two Lie superalgebras M and P with an action of P on M, we can define the *semidirect product* $M \rtimes P$ with the underlying supermodule $M \oplus P$ endowed with the bracket given by

$$[(m,p),(m',p')] = ([m,m'] + {}^{p}m' - (-1)^{|m||p'|}({}^{p'}m),[p,p']).$$

Now we are ready to introduce the following notion of crossed modules of Lie superalgebras (see also [23, Definition 5]).

Definition 2.2.6. A *crossed module of Lie superalgebras* is a homomorphism of Lie superalgebras $\partial \colon M \to P$ with an action of P on M satisfying

(i) $\partial({}^{p}m) = [p,\partial(m)]$,

(ii) ${}^{\partial(m)}m' = [m,m']$,

for all $p \in P$ and $m,m' \in M$.

Example 2.2.7. There are some standard examples of crossed modules:

(i) The inclusion $M \hookrightarrow P$ of a graded ideal M of a Lie superalgebra P is a crossed module of Lie superalgebras.

(ii) If P is a Lie superalgebra and M is a P-supermodule, the trivial map $0\colon M \to P$ is a crossed module of Lie superalgebras.

(iii) A central extension of Lie superalgebras $\partial\colon M \twoheadrightarrow P$ (i.e., $\mathsf{Ker}\,\partial \subseteq \mathsf{Z}(M)$) is a crossed module of Lie superalgebras. Here the action of P on M is given by ${}^{p}m = [\bar{m}, m]$, where $\bar{m} \in M$ is any element of $\partial^{-1}(p)$.

(iv) The homomorphism of Lie superalgebras $\partial\colon M \to \mathrm{Der}_{\mathbb{K}}(M)$ which sends $m \in M$ to the inner derivation $\mathsf{ad}(m) \in \mathrm{Der}_{\mathbb{K}}(M)$, defined by $\mathsf{ad}(m)(m') = [m, m']$, together with the action of $\mathrm{Der}_{\mathbb{K}}(M)$ on M given by ${}^{D}m = D(m)$, is a crossed module of Lie superalgebras.

Lemma 2.2.8. *Let $\partial\colon M \to P$ be a crossed module of Lie superalgebras. Then the following conditions are satisfied:*

(i) *The kernel of ∂ is in the centre of M.*

(ii) *The image of ∂ is a graded ideal of P.*

(iii) *The Lie superalgebra $\mathsf{Im}\,\partial$ acts trivially on the centre $\mathsf{Z}(M)$, and so trivially on $\mathsf{Ker}\,\partial$. Hence $\mathsf{Ker}\,\partial$ inherits an action of $P/\mathsf{Im}\,\partial$ making $\mathsf{Ker}\,\partial$ a $P/\mathsf{Im}\,\partial$-supermodule.*

Proof. This is an immediate consequence of Definition 2.2.6. □

2.2.3 Free Lie superalgebra and enveloping superalgebra of a Lie superalgebra

Definition 2.2.9. The *free Lie superalgebra* on a $\mathbb{Z}_2$-graded set $X = X_{\bar{0}} \cup X_{\bar{1}}$ is a Lie superalgebra $\mathrm{F}(X)$ together with a degree zero map $i\colon X \to \mathrm{F}(X)$ such that if M is any Lie superalgebra and $j\colon X \to M$ is a degree zero map, then there is a unique Lie superalgebra homomorphism $h\colon \mathrm{F}(X) \to M$ with $j = h \circ i$.

The existence of free Lie superalgebras is guaranteed by an analogue of Witt's theorem (see [18, Theorem 6.2.1]). In the sequel we need the following construction of the free Lie superalgebra.

Construction 2.2.10. *Let* $X = X_{\bar{0}} \cup X_{\bar{1}}$ *be a* $\mathbb{Z}_2$*-graded set. Denote by* $\mathrm{mag}(X)$ *the free magma over the set* X*. The free superalgebra on* X*, denoted by* $\mathrm{alg}(X)$*, has as elements the finite sums* $\sum_i \lambda_i x_i$*, where* $\lambda_i \in \mathbb{K}$ *and* x_i *are elements of* $\mathrm{mag}(X)$ *and the multiplication in* $\mathrm{alg}(X)$ *extends the multiplication in* $\mathrm{mag}(X)$*. Note that the grading is naturally defined in* $\mathrm{alg}(X)$*. The free Lie superalgebra* $\mathrm{F}(X)$ *is the quotient* $\mathrm{alg}(X)/I$*, where* I *is the graded ideal generated by the elements*

$$xy + (-1)^{|x||y|}yx,$$
$$(-1)^{|x||z|}\big(x(yz)\big) + (-1)^{|y||x|}\big(y(zx)\big) + (-1)^{|z||y|}\big(z(xy)\big),$$
$$x_{\bar{0}}x_{\bar{0}},$$

for all homogeneous $x, y, z \in X$ *and* $x_{\bar{0}} \in X_{\bar{0}}$*.*

Definition 2.2.11. The *universal enveloping superalgebra* of a Lie superalgebra M is a pair $(\mathrm{U}(M), \sigma)$, where $\mathrm{U}(M)$ is a unital associative superalgebra and $\sigma\colon M \to \mathrm{U}(M)$ is an even linear map satisfying

$$\sigma[x, y] = \sigma(x)\sigma(y) - (-1)^{|x||y|}\sigma(y)\sigma(x), \tag{2.2.1}$$

for all homogeneous $x, y \in M$, such that the following universal property holds: for any other pair (A, σ'), where A is a unital associative superalgebra and $\sigma'\colon M \to A$ is an even linear map satisfying (2.2.1), there is a unique superalgebra homomorphism $f\colon \mathrm{U}(M) \to A$ such that $f \circ \sigma = \sigma'$.

Now we need to recall (see e. g. [21]) that, given two supermodules M and N, the tensor product of modules $M \otimes_{\mathbb{K}} N$ has a natural supermodule structure with $\mathbb{Z}_2$-grading given by

$$(M \otimes_{\mathbb{K}} N)_{\bar{\alpha}} = \bigoplus_{\bar{\beta}+\bar{\gamma}=\bar{\alpha}} (M_{\bar{\beta}} \otimes_{\mathbb{K}} N_{\bar{\gamma}}).$$

In particular, the tensor power $M^{\otimes n}$, $n \geqslant 2$, has the induced $\mathbb{Z}_2$-grading. Hence the tensor algebra $T(M)$ has the $\mathbb{Z}_2$-grading extending that of M. We call $T(M)$ *the tensor superalgebra.*

Construction 2.2.12. *Let* M *be a Lie superalgebra and* $T(M)$ *the tensor superalgebra over the underlying supermodule of* M*. Consider the two-sided ideal* $J(M)$ *of* $T(M)$ *generated by all elements of the form*

$$m \otimes m' - (-1)^{|m||m'|}m' \otimes m - [m, m'],$$

for all homogeneous $m, m' \in M$. Then the quotient $\mathrm{U}(M) = T(M)/J(M)$ *is a unital associative superalgebra. By composing the canonical inclusion* $M \to T(M)$ *with the canonical projection* $T(M) \to \mathrm{U}(M)$ *we get the canonical even linear map* $\sigma\colon M \to \mathrm{U}(M)$*. Then the pair* $(\mathrm{U}(M), \sigma)$ *is the universal enveloping superalgebra of* M *(see [2]).*

Note that, as in the Lie algebra case, the universal enveloping superalgebra turns out to be a very useful tool for the representation theory of Lie superalgebras. In particular, by the universal property, it follows that a Lie supermodule over a Lie superalgebra M is the same as a $\mathbb{Z}_2$-graded (left) $\mathrm{U}(M)$-module (see [21, Chapter 1]).

Let us consider $\mathbb{K}$ with $\mathbb{Z}_2$-grading concentrated in degree zero, that is, with $\mathbb{K}_{\bar{1}} = 0$. Then the trivial map from a Lie superalgebra M into $\mathbb{K}$ gives rise to a unique homomorphism of superalgebras $\varepsilon\colon \mathrm{U}(M) \to \mathbb{K}$. The kernel of ε, denoted by $\Omega(M)$, is called the *augmentation ideal* of M. Obviously, $\Omega(M)$ is just the graded ideal of $\mathrm{U}(M)$ generated by $\sigma(M)$.

2.2.4 Homology of Lie superalgebras

Now we briefly recall from [18, 22] the definition of homology of Lie superalgebras.

The *Grassmann algebra* of a Lie superalgebra P, denoted by $\bigwedge_{\mathbb{K}}(P)$, is defined to be the quotient of the tensor superalgebra $T(P)$ of P by the ideal generated by the elements

$$x \otimes y + (-1)^{|x||y|} y \otimes x,$$

for all homogeneous $x, y \in P$. Note that $\bigwedge_{\mathbb{K}}(P) = \bigoplus_{n>0} \bigwedge_{\mathbb{K}}^n(P)$, where $\bigwedge_{\mathbb{K}}^n(P)$ is the image of $P^{\otimes n}$ in $\bigwedge_{\mathbb{K}}(P)$, has an induced P-supermodule structure given by

$$x(x_1 \wedge \cdots \wedge x_n) = \sum_{i=1}^{n} (-1)^{|x| \sum_{k<i} |x_k|} (x_1 \wedge \cdots \wedge [x, x_i] \wedge \cdots \wedge x_n).$$

Let M be a P-supermodule and consider the chain complex $(C_*(P, M), d_*)$ defined by $C_n(P, M) = \bigwedge_{\mathbb{K}}^n(P) \otimes_{\mathbb{K}} M$, for $n \geqslant 0$, with boundary maps

$d_n\colon C_n(P,M) \to C_{n-1}(P,M)$ defined on generators by

$$d_n(x_1 \wedge \cdots \wedge x_n \otimes y) = \sum_{i=1}^{n} (-1)^{i+|x_i|\sum_{k>i}|x_k|}(x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_n \otimes x_i y) + \sum_{i<j} (-1)^{\omega}([x_i, x_j] \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_n \otimes y),$$

where $\omega = i + j + |x_i| \sum_{k<i} |x_k| + |x_j| \sum_{l<j} |x_l| + |x_i||x_j|$. The n-th *homology of the Lie superalgebra* P with coefficients in the P-supermodule M, $\mathrm{H}_n(P,M)$, is the n-th homology of the chain complex $(C_*(P,M), d_*)$, i.e.

$$\mathrm{H}_n(P,M) = \frac{\mathsf{Ker}\, d_n}{\mathsf{Im}\, d_{n+1}}.$$

If $\mathbb{K}$ is regarded as a trivial P-supermodule, we write $\mathrm{H}_n(P)$ for $\mathrm{H}_n(P,\mathbb{K})$.

In the case when the ground ring $\mathbb{K}$ is a field, there is a relation between Tor functor and the homology (see [18]) given by

$$\mathrm{H}_n(P,M) \cong \mathsf{Tor}_n^{\mathrm{U}(P)}(\mathbb{K}, M).$$

By analogy to Lie algebras (see e. g. [11]), we have the following isomorphisms

$$H_0(P,M) \cong \mathrm{Coker}\,\big(\Omega(P) \otimes_{\mathrm{U}(P)} M \longrightarrow M\big), \tag{2.2.2}$$

$$H_1(P,M) \cong \mathsf{Ker}\,\big(\Omega(P) \otimes_{\mathrm{U}(P)} M \longrightarrow M\big). \tag{2.2.3}$$

2.3 Non-abelian tensor product of Lie superalgebras

In this section we introduce a non-abelian tensor product of Lie superalgebras, which generalizes the non-abelian tensor product of Lie algebras [8], and study its properties.

2.3.1 Construction of the non-abelian tensor product

Definition 2.3.1. Let M and N be two Lie superalgebras with actions on each other. Let $X_{M,N}$ be the $\mathbb{Z}_2$-graded set of all symbols $m \otimes n$, where $m \in M_{\bar{0}} \cup M_{\bar{1}}$,

$n \in N_{\bar{0}} \cup N_{\bar{1}}$ and the $\mathbb{Z}_2$-gradation is given by $|m \otimes n| = |m| + |n|$. We define the *non-abelian tensor product* of M and N, denoted by $M \otimes N$, as the Lie superalgebra generated by $X_{M,N}$ and subject to the relations:

(i) $\lambda(m \otimes n) = \lambda m \otimes n = m \otimes \lambda n$,

(ii) $(m + m') \otimes n = m \otimes n + m' \otimes n,\quad$ where m, m' have the same grade,
$m \otimes (n + n') = m \otimes n + m \otimes n',\quad$ where n, n' have the same grade,

(iii) $[m, m'] \otimes n = m \otimes {}^{m'}n - (-1)^{|m||m'|}(m' \otimes {}^{m}n)$,
$m \otimes [n, n'] = (-1)^{|n'|(|m|+|n|)}({}^{n'}m \otimes n) - (-1)^{|m||n|}({}^{n}m \otimes n')$,

(iv) $[m \otimes n, m' \otimes n'] = -(-1)^{|m||n|}({}^{n}m \otimes {}^{m'}n')$,

for every $\lambda \in \mathbb{K}$, $m, m' \in M_{\bar{0}} \cup M_{\bar{1}}$ and $n, n' \in N_{\bar{0}} \cup N_{\bar{1}}$.

Let us remark that if $m = m_{\bar{0}} + m_{\bar{1}}$ is any element of M and $n = n_{\bar{0}} + n_{\bar{1}}$ is any element of N, then under the notation $m \otimes n$ we mean the sum

$$m_{\bar{0}} \otimes n_{\bar{0}} + m_{\bar{0}} \otimes n_{\bar{1}} + m_{\bar{1}} \otimes n_{\bar{0}} + m_{\bar{1}} \otimes n_{\bar{1}}.$$

If $M = M_{\bar{0}}$ and $N = N_{\bar{0}}$ then $M \otimes N$ is the non-abelian tensor product of Lie algebras introduced and studied in [8] (see also [12]).

Definition 2.3.2. Actions of Lie superalgebras M and N on each other are said to be *compatible* if

(i) ${}^{({}^{n}m)}n' = -(-1)^{|m||n|}[{}^{m}n, n']$,

(ii) ${}^{({}^{m}n)}m' = -(-1)^{|m||n|}[{}^{n}m, m']$,

for all $m, m' \in M_{\bar{0}} \cup M_{\bar{1}}$ and $n, n' \in N_{\bar{0}} \cup N_{\bar{1}}$.

For example, if M and N are two graded ideals of some Lie superalgebra, the actions induced by the bracket are compatible.

Proposition 2.3.3. *Let M and N be Lie superalgebras acting compatibly on each other. Then there is a natural isomorphism of Lie superalgebras*

$$M \otimes N \cong \frac{M \otimes_{\mathbb{K}} N}{D(M, N)},$$

where $D(M, N)$ is the submodule of the supermodule $M \otimes_{\mathbb{K}} N$ generated by the elements

(i) $[m, m'] \otimes n - m \otimes {}^{m'}n + (-1)^{|m||m'|}(m' \otimes {}^{m}n)$,

(ii) $m \otimes [n, n'] - (-1)^{|n'|(|m|+|n|)}({}^{n'}m \otimes n) + (-1)^{|m||n|}({}^{n}m \otimes n')$,

(iii) $({}^{n}m) \otimes ({}^{m}n)$, *with* $|m| = |n|$

(iv) $(-1)^{|m||n|}({}^{n}m) \otimes ({}^{m'}n') + (-1)^{(|m|+|n|)(|m'|+|n'|)+|m'||n'|}({}^{n'}m') \otimes ({}^{m}n)$,

(v) $\displaystyle\circlearrowleft_{(m,n),(m',n'),(m'',n'')} (-1)^{(|m|+|n|)(|m''|+|n''|)+|m||n|+|m'||n'|}[{}^{n}m, {}^{n'}m'] \otimes ({}^{m''}n'')$,

for all $m, m', m'' \in M_{\bar{0}} \cup M_{\bar{1}}$ *and* $n, n', n'' \in N_{\bar{0}} \cup N_{\bar{1}}$, *where* $\circlearrowleft_{x,y,z}$ *denotes the cyclic summation with respect to* x, y, z.

Proof. There is a Lie superalgebra structure on the supermodule $(M \otimes_{\mathbb{K}} N)/D(M, N)$ given on generators by the following bracket

$$[m \otimes n, m' \otimes n'] = -(-1)^{|m||n|}({}^{n}m \otimes {}^{m'}n'),$$

for all $m, m' \in M_{\bar{0}} \cup M_{\bar{1}}$, $n, n' \in N_{\bar{0}} \cup N_{\bar{1}}$ and extended by linearity. It is routine to check that this bracket is compatible with the defining relations of $(M \otimes_{\mathbb{K}} N)/D(M, N)$ and it indeed defines a Lie superalgebra structure. Then the canonical homomorphism $M \otimes N \to (M \otimes_{\mathbb{K}} N)/D(M, N)$, $m \otimes n \mapsto m \otimes n$, is an isomorphism. □

The proof of the following proposition is a routine calculation.

Proposition 2.3.4. *Let* M *and* N *be two Lie superalgebras acting compatibly on each other.*

(i) *The following morphisms*

$$\mu \colon M \otimes N \to M, \quad m \otimes n \mapsto -(-1)^{|m||n|}({}^{n}m),$$
$$\nu \colon M \otimes N \to N, \quad m \otimes n \mapsto {}^{m}n,$$

are Lie superalgebra homomorphisms.

(ii) *There are actions of* M *and* N *on* $M \otimes N$ *given by*

$${}^{m'}(m \otimes n) = [m', m] \otimes n + (-1)^{|m||m'|}m \otimes ({}^{m'}n),$$
$${}^{n'}(m \otimes n) = ({}^{n'}m) \otimes n + (-1)^{|n||n'|}m \otimes [n', n],$$

for $m, m' \in M_{\bar{0}} \cup M_{\bar{1}}$, $n, n' \in N_{\bar{0}} \cup N_{\bar{1}}$ *and extended by linearity. Moreover, with these actions* μ *and* ν *are crossed modules of Lie superalgebras.*

We will denote by $[M,N]^M$ (resp. $[M,N]^N$) the image of μ (resp. ν), which by Lemma 2.2.8(ii) is a graded ideal of M (resp. N) generated by the elements of the form ${}^n m$ (resp. ${}^m n$) for $m \in M$ and $n \in N$. Note that by Lemma 2.2.8(iii) $\mathsf{Ker}(\mu)$ $\big($resp. $\mathsf{Ker}(\nu)\big)$ is an $M/[M,N]^M$-supermodule (resp. $N/[M,N]^N$-supermodule).

2.3.2 Some properties of the non-abelian tensor product

The obvious analogues of Brown and Loday results [1] hold for Lie superalgebras. In the following two propositions immediately below we show that sometimes the non-abelian tensor product of Lie superalgebras can be expressed in terms of the tensor product of supermodules.

Proposition 2.3.5. *Let M and N be Lie superalgebras acting on each other. Then the canonical map $M \otimes_{\mathbb{K}} N \to M \otimes N$, $m \otimes n \mapsto m \otimes n$, is an even, surjective homomorphism of supermodules. In addition, if M and N act trivially on each other, then $M \otimes N$ is an abelian Lie superalgebra and there is an isomorphism of supermodules*

$$M \otimes N \cong M^{\mathrm{ab}} \otimes_{\mathbb{K}} N^{\mathrm{ab}},$$

where $M^{\mathrm{ab}} = M/[M,M]$ and $N^{\mathrm{ab}} = N/[N,N]$.

Proof. It is straightforward by the identities (iv), (iii) of Definition 2.3.1. □

Proposition 2.3.6. *Let P be a Lie superalgebra and M a P-supermodule considered as an abelian Lie superalgebra acting trivially on P. Then there is an isomorphism of supermodules*

$$P \otimes M \cong \Omega(P) \otimes_{\mathrm{U}(P)} M.$$

Proof. By Proposition 2.3.3 there is an isomorphism of supermodules

$$P \otimes M \cong \frac{P \otimes_{\mathbb{K}} M}{W},$$

where W is the submodule of $P \otimes_{\mathbb{K}} M$ generated by all elements of the form

$$[p,p'] \otimes m - p \otimes p'm + (-1)^{|p||p'|} p' \otimes pm$$

for all $p, p' \in P_{\bar{0}} \cup P_{\bar{1}}$ and $m \in M_{\bar{0}} \cup M_{\bar{1}}$. Now by using Construction 2.2.12 and by repeating the respective part of the proof of [3, Proposition 13], it is easy to see that there is an isomorphism of supermodules

$$\frac{P \otimes_{\mathbb{K}} M}{W} \cong \Omega(P) \otimes_{\mathrm{U}(P)} M,$$

which completes the proof. □

The non-abelian tensor product of Lie superalgebras is symmetric, in the sense of the following proposition.

Proposition 2.3.7. *The Lie superalgebra homomorphism*

$$M \otimes N \to N \otimes M, \quad m \otimes n \mapsto -(-1)^{|m||n|}(n \otimes m),$$

is an isomorphism.

Proof. This can be checked readily. □

Let us consider the category $\mathtt{SLie}^2_{\mathbb{K}}$ whose objects are ordered pairs of Lie superalgebras (M, N) acting compatibly on each other, and the morphisms are pairs of Lie superalgebra homomorphisms $(\phi\colon M \to M', \psi\colon N \to N')$ which preserve the actions, i.e., $\phi({}^n m) = {}^{\psi(n)}\phi(m)$ and $\psi({}^m n) = {}^{\phi(m)}\psi(n)$. For such a pair (ϕ, ψ) we have a homomorphism of Lie superalgebras $\phi \otimes \psi\colon M \otimes N \to M' \otimes N'$, $m \otimes n \mapsto \phi(m) \otimes \psi(n)$. Therefore, $\otimes$ is a functor from $\mathtt{SLie}^2_{\mathbb{K}}$ to the category of Lie superalgebras.

Given an exact sequence in $\mathtt{SLie}^2_{\mathbb{K}}$

$$(0,0) \longrightarrow (K, L) \xrightarrow{(i,j)} (M, N) \xrightarrow{(\phi,\psi)} (P, Q) \longrightarrow (0,0), \tag{2.3.1}$$

by Proposition 2.3.4(ii) there is a Lie superalgebra homomorphism $M \otimes L \to L$ and an action of N on $K \otimes N$. Thus, there is an action of $M \otimes L$ on $K \otimes N$, so we can form the semidirect product $(K \otimes N) \rtimes (M \otimes L)$, and we have the following obvious analogue of [8, Proposition 9].

Proposition 2.3.8. *Given the short exact sequence* (2.3.1)*, there is an exact sequence of Lie superalgebras*

$$(K \otimes N) \rtimes (M \otimes L) \xrightarrow{\alpha} M \otimes N \xrightarrow{\phi\otimes\psi} P \otimes Q \longrightarrow 0.$$

In particular, given a Lie superalgebra M and a graded ideal K of M, there is an exact sequences of Lie superalgebras

$$(K \otimes M) \rtimes (M \otimes K) \to M \otimes M \to (M/K) \otimes (M/K) \to 0. \tag{2.3.2}$$

2.3.3 Nilpotency, solvability and Engel of the non-abelian tensor product

The results from [20] on nilpotency, solvability and Engel of the non-abelian tensor product on Lie algebras can be easily extended to the case of Lie superalgebras. The notions of nilpotency and solvability of Lie superalgebras are given in [18]. As they are very similar to the respective notions for Lie algebras, we omit them. We say that a Lie superalgebra M is *n-Engel* if it satisfies $\mathsf{ad}(x)^n = 0$ for all $x \in M$. The proof of the following result is similar to the proof of [20, Theorem 2.2].

Theorem 2.3.9. *Let M and N be two Lie superalgebras acting compatibly on each other. Then,*

(i) *If $[M,N]^M$ is nilpotent, then $M \otimes N$ and $[M,N]^N$ are nilpotent too. Moreover, if the nilpotency class of $[M,N]^M$ is $\mathrm{cl}([M,N]^M)$, then*

$$\begin{aligned}\mathrm{cl}([M,N]^M) \leqslant \mathrm{cl}(M \otimes N) &\leqslant \mathrm{cl}([M,N]^M) + 1,\\ \mathrm{cl}([M,N]^N) &\leqslant \mathrm{cl}([M,N]^M) + 1.\end{aligned}$$

(ii) *If $[M,N]^M$ is solvable, then $M \otimes N$ and $[M,N]^N$ are solvable too. Moreover, if the derived length of $[M,N]^M$ is $\ell([M,N]^M)$, then*

$$\begin{aligned}\ell([M,N]^M) \leqslant \ell(M \otimes N) &\leqslant \ell([M,N]^M) + 1,\\ \ell([M,N]^N) &\leqslant \ell([M,N]^M) + 1.\end{aligned}$$

(iii) *If $[M,N]^M$ is Engel, then $M \otimes N$ and $[M,N]^N$ are Engel too. Moreover, if $[M,N]^M$ is n-Engel, then $M \otimes N$ and $[M,N]^N$ are $(n+1)$-Engel.*

2.4 Universal central extensions of Lie superalgebras

Now we use the non-abelian tensor product of Lie superalgebras to describe universal central extensions of Lie superalgebras. Recall that a central extension $\mathfrak{u}\colon U \twoheadrightarrow P$ is universal if for any other central extension $f\colon M \twoheadrightarrow P$ there is a unique homomorphism $\theta\colon U \to M$ such that $f \circ \theta = \mathfrak{u}$. It is shown in [19] that a Lie superalgebra P admits a universal central extension if and only if P is perfect, i.e. $P = [P,P]$.

It follows from Proposition 2.3.4 and Lemma 2.2.8(i) that the homomorphism $\mathfrak{u}\colon P \otimes P \twoheadrightarrow [P, P]$, $\mathfrak{u}(p \otimes p') = [p, p']$, is a central extension of the Lie superalgebra $[P, P]$.

Theorem 2.4.1. *If P is a perfect Lie superalgebra, then the central extension $\mathfrak{u}\colon P \otimes P \twoheadrightarrow P$ is the universal central extension.*

Proof. Let $f\colon M \twoheadrightarrow P$ be a central extension of P. Since $\mathsf{Ker}\, f$ is in the centre of M, we get a well-defined homomorphism of Lie superalgebras $\theta\colon P \otimes P \to M$ given by $\theta(p \otimes p') = [m_p, m_{p'}]$, where m_p and $m_{p'}$ are any preimages of p and p', respectively. Obviously $\theta \circ f = \mathfrak{u}$. Since P is perfect, then by relation (iv) of Definition 2.3.1, so is $P \otimes P$. Then by [19, Lemma 1.4] the homomorphism θ is unique. □

Remark 2.4.2. If P is a perfect Lie superalgebra, then $\mathrm{H}_2(P) \approx \mathsf{Ker}(P \otimes P \xrightarrow{\mathfrak{u}} P)$, since the kernel of the universal central extension is isomorphic to the second homology $\mathrm{H}_2(P)$ (see [19]).

It is a classical result that the universal central extension of the Lie algebra $\mathfrak{sl}(n, A)$, where A is a unital associative algebra, is the Steinberg algebra $\mathfrak{st}(n, A)$, when $n \geqslant 5$ (see e. g. [16]). Recently, in [5, 9], this result has been extended to Lie superalgebras. Below, using the non-abelian tensor product of Lie superalgebras, we propose an alternative proof of the same result.

First we recall from [5] that, given a unital associative superalgebra A, the *Lie superalgebra* $\mathfrak{sl}(m, n, A)$, $m + n \geqslant 3$, is defined to be the subalgebra of the Lie superalgebra $\mathfrak{gl}(m, n, A)$ (see Example 2.2.2 (ii)) generated by the elements $E_{ij}(a)$, $1 \leqslant i \neq j \leqslant m+n$, $a \in A_{\bar{0}} \cup A_{\bar{1}}$. It is shown in [5, Lemma 3.3] that $\mathfrak{sl}(m, n, A)$ is a perfect Lie superalgebra. This guarantees the existence of the universal central extension of $\mathfrak{sl}(m, n, A)$.

The Steinberg Lie superalgebra $\mathfrak{st}(m, n, A)$ is defined for $m + n \geqslant 3$ to be the Lie superalgebra generated by the homogeneous elements $F_{ij}(a)$, where $1 \leqslant i \neq j \leqslant m + n$, $a \in A$ is a homogeneous element and the $\mathbb{Z}_2$-grading is given by $|F_{ij}(a)| = |i| + |j| + |a|$, subject to the following relations:

$$
\begin{aligned}
&a \mapsto F_{ij}(a) \text{ is a } \mathbb{K}\text{-linear map},\\
&[F_{ij}(a), F_{jk}(b)] = F_{ik}(ab), \text{ for distinct } i, j, k,\\
&[F_{ij}(a), F_{kl}(b)] = 0, \text{ for } j \neq k, i \neq l.
\end{aligned}
$$

Theorem 2.4.3 ([5]). *If $m + n \geqslant 5$, then the canonical epimorphism*

$$\mathfrak{st}(m,n,A) \twoheadrightarrow \mathfrak{sl}(m,n,A), \quad F_{ij}(a) \mapsto E_{ij}(a),$$

is the universal central extension of the perfect Lie superalgebra $\mathfrak{sl}(m,n,A)$.

Proof. We claim that there is an isomorphism of Lie superalgebras

$$\mathfrak{st}(m,n,A) \cong \mathfrak{st}(m,n,A) \otimes \mathfrak{st}(m,n,A).$$

Indeed, one can readily check that the maps

$$\mathfrak{st}(m,n,A) \longrightarrow \mathfrak{st}(m,n,A) \otimes \mathfrak{st}(m,n,A), F_{ij}(a) \mapsto F_{ik}(a) \otimes F_{kj}(1) \text{ for } k \neq i,j,$$
$$\mathfrak{st}(m,n,A) \otimes \mathfrak{st}(m,n,A) \longrightarrow \mathfrak{st}(m,n,A), F_{ij}(a) \otimes F_{kl}(b) \mapsto [F_{ij}(a), F_{kl}(b)],$$

are well-defined homomorphisms of Lie superalgebras if $m + n \geqslant 5$, and they are inverses to each other. Since $\mathfrak{st}(m,n,A)$ is a perfect Lie superalgebra, then Theorem 2.4.1 and [19, Corollary 1.9] complete the proof. □

2.5 Non-abelian homology of Lie superalgebras

The low-dimensional non-abelian homology of Lie algebras with coefficients in crossed modules was defined in [10] and it was extended to all dimensions in [12]. In this section we extend to Lie superalgebras the construction of zero and first non-abelian homologies. We also relate the non-abelian homology of Lie superalgebras with the cyclic homology of associative superalgebras studied in [13, 15].

2.5.1 Construction of the non-abelian homology and some properties

Let P be a Lie superalgebra. We denote by `Cross`(P) the category of crossed modules of Lie superalgebras over P (crossed P-modules, for short), whose objects are crossed modules $(M, \partial) \equiv (\partial \colon M \to P)$ and a morphism from (M, ∂) to (N, ∂') is a Lie superalgebra homomorphism $f \colon M \to N$ such that $f({}^p m) = {}^p f(m)$ for all $p \in P$, $m \in M$ and $\partial' \circ f = \partial$. By an exact sequence $(L, \partial'') \xrightarrow{f} (M, \partial) \xrightarrow{g} (N, \partial')$ in `Cross`(P) we mean that the sequence of Lie superalgebras $L \xrightarrow{f} M \xrightarrow{g} N$ is exact.

Lemma 2.5.1. *Given a short exact sequence in* $\mathcal{Cross}$(P)

$$0 \to (L, \partial'') \xrightarrow{f} (M, \partial) \xrightarrow{g} (N, \partial') \to 0,$$

the morphism $\partial'' \colon L \to P$ *is trivial and* L *is an abelian Lie superalgebra.*

Proof. Clearly $\partial'' = \partial' \circ g \circ f = 0$ and $[l, l'] = {}^{\partial''(l)}l' = 0$, for all $l, l' \in L$. □

If (M, ∂) and (N, ∂') are two crossed P-modules, then the Lie superalgebras M and N act compatibly on each other via the action of P. Thus, we can construct the non-abelian tensor product of Lie superalgebras $M \otimes N$. Moreover, we have an action of P on $M \otimes N$ defined by ${}^{p}(m \otimes n) = {}^{p}m \otimes n + (-1)^{|p||m|} m \otimes {}^{p}n$, and straightforward computations show that $\eta \colon M \otimes N \to P$, $m \otimes n \mapsto [\partial(m), \partial'(n)]$, is a crossed P-module.

Proposition 2.5.2. *Let* (M, ∂) *be a crossed* P*-module. There is a right exact functor* $(M \otimes -) \colon \mathcal{Cross}(\mathrm{P}) \to \mathcal{Cross}(\mathrm{P})$ *given, for any crossed* P*-module* (N, ∂')*, by*

$$(M \otimes -)(N, \partial') = (M \otimes N, \eta).$$

Proof. It is an immediate consequence of Proposition 2.3.8 □

Definition 2.5.3. Let (M, ∂) be a crossed P-module. We define the zero and first non-abelian homologies of P with coefficients in M by setting

$$\mathcal{H}_0(P, M) = \operatorname{Coker} \nu \qquad \text{and} \qquad \mathcal{H}_1(P, M) = \mathsf{Ker}\, \nu,$$

where $\nu \colon P \otimes M \to M$, $p \otimes m \mapsto {}^{p}m$, is the Lie superalgebra homomorphism as in Proposition 2.3.4.

If we consider the crossed P-module (P, id_P) we have that

$$\mathcal{H}_0(P, P) = \frac{P}{[P, P]} \cong \mathrm{H}_1(P).$$

In addition, if P is perfect, by Theorem 2.4.1 we have that $\mathcal{H}_1(P, P) \cong \mathrm{H}_2(P)$.

The zero and first non-abelian homologies generalize respectively the zero and first homologies of Lie superalgebras in the sense of the following proposition.

Proposition 2.5.4. *Let the ground ring $\mathbb{K}$ be a field. Let P be a Lie superalgebra and M a P-supermodule thought as a crossed P-module $(M, 0)$. Then there are isomorphisms of super vector spaces*

$$\mathcal{H}_0(P, M) \cong \mathrm{H}_0(P, M) \quad \text{and} \quad \mathcal{H}_1(P, M) \cong \mathrm{H}_1(P, M).$$

Proof. This is a direct consequence of Proposition 2.3.6 and the isomorphisms (2.2.2) and (2.2.3). □

Proposition 2.5.5. *Given a short exact sequence in* Cross(P)

$$0 \to (L, 0) \to (M, \partial) \to (N, \partial') \to 0$$

we have an exact sequence of supermodules

$$\mathcal{H}_1(P, L) \to \mathcal{H}_1(P, M) \to \mathcal{H}_1(P, N) \to \mathcal{H}_0(P, L) \to \mathcal{H}_0(P, M) \to \mathcal{H}_0(P, N) \to 0.$$

Proof. The proof is an immediate consequence of the snake lemma applied to the diagram obtained from Proposition 2.5.2

$$\begin{array}{ccccccccc} & & P \otimes L & \longrightarrow & P \otimes M & \longrightarrow & P \otimes N & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0. \end{array}$$

□

2.5.2 Application to the cyclic homology of associative superalgebras

Now we recall from [15] and [13] the definition of cyclic homology of associative superalgebras. Let A be an associative superalgebra and $(C'_*(A), d'_*)$ denote its Hochschild complex, that is $C'_n(A) = A^{\otimes_{\mathbb{K}}(n+1)}$ and the boundary map $d_n \colon C'_n(A) \to C'_{n-1}(A)$ is given by

$$d'_n(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n + (-1)^{n+|a_n|(|a_0|+\cdots+|a_{n-1}|)} a_n a_0 \otimes \cdots \otimes a_{n-1}.$$

Now the cyclic group $\mathbb{Z}/(n+1)\mathbb{Z}$ acts on $A^{\otimes_{\mathbb{K}}(n+1)}$ via

$$t_n(a_0 \otimes \cdots \otimes a_n) = (-1)^{n+|a_n|\sum_{k<n}|a_k|} a_n \otimes a_0 \otimes \cdots \otimes a_{n-1},$$

where $t_n = 1 + (n+1)\mathbb{Z} \in \mathbb{Z}/(n+1)\mathbb{Z}$. For each $n \geqslant 0$, consider the quotient $C_n(A) = A^{\otimes_{\mathbb{K}}(n+1)}/\operatorname{Im}(1-t_n)$ which is the module of coinvariants of $C'_n(A)$ under the $\mathbb{Z}/(n+1)\mathbb{Z}$-action. Then d'_n induces a well-defined map $d_n \colon C_n(A) \to C_{n-1}(A)$ and there is an induced chain complex $(C_*(A), d_*)$, which is called the Connes complex of A. Its homologies are, by definition, the *cyclic homologies of the associative superalgebra* A, denoted by $\mathrm{HC}_n(A)$, $n \geqslant 0$.

Easy calculations show that, given an associative superalgebra A, $\mathrm{HC}_1(A)$ is the kernel of the homomorphism of supermodules

$$(A \otimes_{\mathbb{K}} A)/\mathrm{I}(A) \to [A, A], \quad a \otimes b \mapsto ab - (-1)^{|a||b|} ba,$$

where $[A, A]$ is the graded submodule of A generated by the elements $ab - (-1)^{|a||b|} ba$ and $\mathrm{I}(A)$ is the graded submodule of the supermodule $A \otimes_{\mathbb{K}} A$ generated by the elements

$$a \otimes b + (-1)^{|a||b|} b \otimes a,$$
$$ab \otimes c - a \otimes bc + (-1)^{|c|(|a|+|b|)} ca \otimes b,$$

for all homogeneous $a, b, c \in A$.

Now let us consider A as a Lie superalgebra (see Example 2.2.2(i)). Then there is a Lie superalgebra structure on $(A \otimes_{\mathbb{K}} A)/\mathrm{I}(A)$ given by

$$[a \otimes b, a' \otimes b'] = [a, b] \otimes [a', b']$$

for all $a, a', b, b' \in A$. We denote this Lie superalgebra by $\mathrm{V}(A)$. In fact, $\mathrm{V}(A)$ is the quotient of the non-abelian tensor product $A \otimes A$ by the graded ideal generated by the elements $x \otimes y + (-1)^{|x||y|} y \otimes x$ and $xy \otimes z - x \otimes yz + (-1)^{|z|(|x|+|y|)} zx \otimes y$, for all homogeneous $x, y, z \in A$.

Proposition 2.5.6. *Let A be a Lie superalgebra. Then the following assertions hold:*

(i) *There are compatible actions of the Lie superalgebras A and $\mathrm{V}(A)$ on each other.*

(ii) *The map $\mu \colon \mathrm{V}(A) \to A$ given by $x \otimes y \mapsto [x, y]$, together with the action of A on $\mathrm{V}(A)$, is a crossed module of Lie superalgebras.*

(iii) *The action of A on* $\mathrm{V}(A)$ *induces the trivial action of A on* $\mathrm{HC}_1(A)$.

(iv) *There is a short exact sequence in the category* ***Cross***(A)

$$0 \to (\mathrm{HC}_1(A), 0) \to (\mathrm{V}(A), \mu) \to ([A, A], i) \to 0,$$

where $i\colon [A, A] \to A$ *is the inclusion.*

Proof.

(i) The action of A on $\mathrm{V}(A)$ is induced by the action of A on $A \otimes A$ given in Proposition 2.3.4(ii), that is

$$\begin{aligned} {}^{a}(x \otimes y) &= [a, x] \otimes y + (-1)^{|a||x|} x \otimes [a, y] \\ &= ax \otimes y + (-1)^{|a|(|x|+|y|)} x \otimes ya - (-1)^{|x||a|} x \otimes ay - (-1)^{|x||a|} xa \otimes y \\ &= a \otimes xy - (-1)^{|x||y|} a \otimes yx \\ &= a \otimes [x, y], \end{aligned}$$

whilst the action of $\mathrm{V}(A)$ on A is defined by

$${}^{x \otimes y} a = \big[[x, y], a\big]$$

for all homogeneous $a, x, y \in A$. Straightforward calculations show that these are indeed (compatible) actions of Lie superalgebras.

(ii) Since the crossed module of Lie superalgebras $A \otimes A \to A$, $x \otimes y \mapsto [x, y]$, given in Proposition 2.3.4, vanishes on the elements of the form $x \otimes y + (-1)^{|x||y|} y \otimes x$ and $xy \otimes z - x \otimes yz + (-1)^{|z|(|x|+|y|)} zx \otimes y$, then μ is well defined and obviously it is a crossed module of Lie superalgebras.

(iii) If $\sum_i \lambda_i (x_i \otimes y_i) \in \mathrm{HC}_1(A)$, i.e. $\sum_i \lambda_i [x_i, y_i] = 0$, then for all $a \in A$ we have

$${}^{a}\Big(\sum_i \lambda_i (x_i \otimes y_i)\Big) = \sum_i \lambda_i (a \otimes [x_i, y_i]) = a \otimes \sum_i \lambda_i [x_i, y_i] = 0.$$

(iv) This is an immediate consequence of the assertions above. □

By Proposition 2.5.5 we have the following exact sequence of supermodules

$$
\begin{array}{c}
\mathcal{H}_1\big(A,\mathrm{HC}_1(A)\big) \longrightarrow \mathcal{H}_1\big(A,\mathrm{V}(A)\big) \longrightarrow \mathcal{H}_1(A,[A,A]) \longrightarrow \\
\mathcal{H}_0\big(A,\mathrm{HC}_1(A)\big) \longrightarrow \mathcal{H}_0\big(A,\mathrm{V}(A)\big) \longrightarrow \mathcal{H}_0(A,[A,A]) \longrightarrow 0.
\end{array}
\tag{2.5.1}
$$

Below, we will calculate some of the terms of this exact sequence. At first, by analogy to the Dennis-Stein generators [6], we give a definition of the first Milnor cyclic homology for associative superalgebras.

Definition 2.5.7. Let A be an associative superalgebra. We define the first Milnor cyclic homology $\mathrm{HC}_1^M(A)$ of A to be the quotient of the supermodule $A \otimes_{\mathbb{K}} A$ by the graded ideal generated by the elements

$$
\begin{aligned}
& a \otimes b + (-1)^{|a||b|} b \otimes a, \\
& ab \otimes c - a \otimes bc + (-1)^{|c|(|a|+|b|)} ca \otimes b, \\
& a \otimes bc - (-1)^{|b||c|} a \otimes cb,
\end{aligned}
$$

for all homogeneous $a, b, c \in A$.

It is clear that if A is supercommutative, that is, $ab = (-1)^{|a||b|} ba$, for all homogeneous $a, b \in A$, then $\mathrm{HC}_1(A) \cong \mathrm{HC}_1^M(A)$.

Lemma 2.5.8. *We have the following equalities and isomorphisms*

(i) $\mathcal{H}_0\big(A,\mathrm{HC}_1(A)\big) = \mathrm{HC}_1(A)$,

(ii) $\mathcal{H}_1\big(A,\mathrm{HC}_1(A)\big) \cong A/[A,A] \otimes_{\mathbb{K}} \mathrm{HC}_1(A)$,

(iii) $\mathcal{H}_0(A,[A,A]) = [A,A]/\big[A,[A,A]\big]$,

(iv) $\mathcal{H}_0\big(A,\mathrm{V}(A)\big) \cong \mathrm{HC}_1^M(A)$.

Proof.

(i) Since A acts trivially on $\mathrm{HC}_1(A)$, we have that $\mathrm{Coker}\big(A \otimes \mathrm{HC}_1(A) \to \mathrm{HC}_1(A)\big) = \mathrm{HC}_1(A)$.

(ii) Since $\mathrm{HC}_1(A)$ is abelian, by Proposition 2.3.5 we have that $\mathsf{Ker}\big(A \otimes \mathrm{HC}_1(A) \to \mathrm{HC}_1(A)\big) \cong A/[A,A] \otimes_{\mathbb{K}} \mathrm{HC}_1(A)$.

(iii) and (iv) are straightforward. □

It follows that the exact sequence (2.5.1) can be written as in the following theorem.

Theorem 2.5.9. *If A is a unital associative superalgebra. Then there is an exact sequence of supermodules*

$$\frac{A}{[A,A]} \otimes_{\mathbb{K}} \mathrm{HC}_1(A) \longrightarrow \mathcal{H}_1\big(A, \mathrm{V}(A)\big) \longrightarrow \mathcal{H}_1(A,[A,A]) \longrightarrow \mathrm{HC}_1(A) \longrightarrow \mathrm{HC}_1^M(A) \longrightarrow \frac{[A,A]}{\big[A,[A,A]\big]} \longrightarrow 0.$$

Corollary 2.5.10. *If A is perfect as a Lie superalgebra, we have an exact sequence*

$$0 \to \mathcal{H}_1\big(A, \mathrm{V}(A)\big) \to \mathrm{H}_2(A) \to \mathrm{HC}_1(A) \to 0,$$

where $\mathrm{H}_2(A)$ is the usual second homology of the Lie superalgebra A. If in addition $\mathrm{H}_2(A) = 0$, then all terms of the exact sequence in the previous theorem are trivial.

Proof. Since A is perfect we know that $\mathcal{H}_1(A,A) \cong \mathrm{H}_2(A)$, $A/[A,A] \otimes_{\mathbb{K}} \mathrm{HC}_1(A) = 0$ and the map $A \otimes \mathrm{V}(A) \to \mathrm{V}(A)$ is surjective. □

2.6 Non-abelian exterior product of Lie superalgebras

In this section we extend to Lie superalgebras the definition of the non-abelian exterior product of Lie algebras introduced in [8]. Then we use it to derive the Hopf formula for the second homology of a Lie superalgebra and to construct a six-term exact homology sequence of Lie superalgebras.

2.6.1 Construction of the non-abelian exterior product

Let P be a Lie superalgebra and (M, ∂) and (N, ∂') two crossed P-modules. We consider the actions of M and N on each other via P.

Lemma 2.6.1. *Let $M \square N$ be the graded submodule of $M \otimes N$ generated by the elements*

(a) $m \otimes n + (-1)^{|m'||n'|} m' \otimes n'$, *where* $\partial(m) = \partial'(n')$ *and* $\partial(m') = \partial'(n)$,

(b) $m_{\bar{0}} \otimes n_{\bar{0}}$, *where* $\partial(m_{\bar{0}}) = \partial'(n_{\bar{0}})$,

with $m, m' \in M_{\bar{0}} \cup M_{\bar{1}}$, $n, n' \in N_{\bar{0}} \cup N_{\bar{1}}$, $m_{\bar{0}} \in M_{\bar{0}}$ *and* $n_{\bar{0}} \in N_{\bar{0}}$. *Then,* $M \square N$ *is a graded ideal in the centre of* $M \otimes N$.

Proof. Given an element $m \otimes n + (-1)^{|m'||n'|} m' \otimes n'$ of the form (a), suppose that $|m'| = |n|$, then we have

$$\begin{aligned}
[x \otimes y, m \otimes n + (-1)^{|m'||n'|} m' \otimes n'] &= -(-1)^{|x||y|} ({}^{y}x) \otimes \big({}^{m}n + (-1)^{|m'||n'|} ({}^{m'}n')\big) \\
&= -(-1)^{|x||y|} ({}^{y}x) \otimes \big({}^{\partial(m)}n + (-1)^{|m'||n'|} ({}^{\partial(m')}n')\big) \\
&= -(-1)^{|x||y|} ({}^{y}x) \otimes \big({}^{\partial'(n')}n + (-1)^{|m'||n'|} ({}^{\partial'(n)}n')\big) \\
&= -(-1)^{|x||y|} ({}^{y}x) \otimes \big([n', n] + (-1)^{|n||n'|} [n, n']\big) \\
&= 0.
\end{aligned}$$

This is also true when $|m'| \neq |n|$. Indeed, if $|m'| \neq |n|$, since ∂, ∂' are even maps, the equality $\partial(m) = \partial'(n')$ holds if and only if $\partial(m) = 0 = \partial'(n')$. Now take an element $m_{\bar{0}} \otimes n_{\bar{0}}$ of the form (b). Then we have

$$\begin{aligned}
[x \otimes y, m_{\bar{0}} \otimes n_{\bar{0}}] &= -(-1)^{|x||y|} ({}^{y}x) \otimes ({}^{m_{\bar{0}}}n_{\bar{0}}) \\
&= -(-1)^{|x||y|} ({}^{y}x) \otimes ({}^{\partial(m_{\bar{0}})}n_{\bar{0}}) \\
&= -(-1)^{|x||y|} ({}^{y}x) \otimes ({}^{\partial(n_{\bar{0}})}n_{\bar{0}}) \\
&= -(-1)^{|x||y|} ({}^{y}x) \otimes [n_{\bar{0}}, n_{\bar{0}}] \\
&= 0,
\end{aligned}$$

for any $x \otimes y \in M \otimes N$. This completes the proof. □

Definition 2.6.2. Let P be a Lie superalgebra and (M, ∂) and (N, ∂') two crossed P-modules. The *non-abelian exterior product* $M \wedge N$ of the Lie superalgebras M and N is defined by

$$M \wedge N = \frac{M \otimes N}{M \square N}.$$

The equivalence class of $m \otimes n$ will be denoted by $m \wedge n$.

Note that if $M = M_{\bar{0}}$ and $N = N_{\bar{0}}$ then $M \wedge N$ coincides with the non-abelian exterior product of Lie algebras [8].

Reviewing Section 2.3, one can easily check that most of results on the non-abelian tensor product are fulfilled for the non-abelian exterior product. In particular, there are homomorphisms of Lie superalgebras $M \wedge N \to M$, $M \wedge N \to N$ and actions of M and N on $M \wedge N$, induced respectively by the homomorphisms and actions given in Proposition 2.3.4. It is also satisfied the isomorphism $M \wedge N \cong N \wedge M$. Further, given a short exact sequence of Lie superalgebras $0 \to K \to M \to P \to 0$, as an exterior analogue of the exact sequence (2.3.2), we get the following exact sequence of Lie superalgebras

$$K \wedge M \to M \wedge M \to P \wedge P \to 0. \tag{2.6.1}$$

Given a Lie superalgebra M, since $\mathsf{id}\colon M \to M$ is a crossed module, we can consider $M \wedge M$. It is the quotient of $M \otimes M$ by the following relations

$$\begin{aligned} m \wedge m' &= -(-1)^{|m||m'|} m' \wedge m, \\ m_{\bar{0}} \wedge m_{\bar{0}} &= 0, \end{aligned}$$

for all $m, m' \in M_{\bar{0}} \cup M_{\bar{1}}$ and $m_{\bar{0}} \in M_{\bar{0}}$. In the particular case when M is perfect, it is easy to see that $M \square M = 0$, so $M \wedge M \cong M \otimes M$ and in Theorem 2.4.1 we can replace $M \otimes M$ by $M \wedge M$.

2.6.2 A six term exact homology sequence

In [7], the non-abelian exterior product of Lie algebras is used to construct a six-term exact sequence of homology of Lie algebras. In this section we will extend these results to Lie superalgebras.

First of all, we prove an analogue of Miller's theorem [17] on free Lie superalgebras extending the similar result obtained in [7] for Lie algebras.

Proposition 2.6.3. *Let $F = \mathrm{F}(X)$ be the free Lie superalgebra on a graded set X. Then the homomorphism $F \wedge F \to F$, $x \wedge y \mapsto xy$ is injective.*

Proof. Let us prove that $[F, F] \cong F \wedge F$. Using the same notations as in Construction 2.2.10, we define a map $\phi\colon \mathrm{alg}(X) * \mathrm{alg}(X) \to F \wedge F$ by $\sum_i \lambda_i x_i y_i \mapsto \sum_i \lambda_i (x_i \wedge y_i)$, where $\mathrm{alg}(X) * \mathrm{alg}(X)$ is the free product of superalgebras. It is easy to see that ϕ is a $\mathbb{K}$-superalgebra homomorphism since $[x \wedge y, x' \wedge y'] = xy \wedge x'y'$. The ideal I is contained in $\mathrm{alg}(X) * \mathrm{alg}(X)$ and by using the defining relations of $F \wedge F$ it is not difficult to check that ϕ vanishes on I. So we have an induced map from $[F,F]$ to $F \wedge F$, which is inverse to the homomorphism $F \wedge F \to [F,F]$, $x \wedge y \mapsto xy$. □

Let P be a Lie superalgebra and take the quotient supermodule $(P \wedge_{\mathbb{K}} P)/\operatorname{Im} d_3$, where $d_3\colon \bigwedge^3_{\mathbb{K}}(P) \to \bigwedge^2_{\mathbb{K}}(P)$ is the boundary map in the homology complex $(C_*(P,\mathbb{K}), d_*)$. Here $\mathbb{K}$ is considered as a trivial P-module. We define a bracket in $(P \wedge_{\mathbb{K}} P)/\operatorname{Im} d_3$ by setting

$$[x \wedge y, x' \wedge y'] = [x,y] \wedge [x',y']$$

for all $x, y \in P$. As a particular case of the exterior analogue of Proposition 2.3.3 we have

Lemma 2.6.4. *There is an isomorphism of Lie superalgebras*

$$\frac{P \wedge_{\mathbb{K}} P}{\operatorname{Im} d_3} \approx P \wedge P.$$

Corollary 2.6.5.

(i) *For any Lie superalgebra P there is an isomorphism of supermodules*

$$H_2(P) \cong \mathsf{Ker}(P \wedge P \to P).$$

(ii) $H_2(F) = 0$ *if F is a free Lie superalgebra.*

(iii) *(Hopf Formula) Given a free presentation $0 \to R \to F \to P \to 0$ of a Lie superalgebra P, there is an isomorphism of supermodules*

$$H_2(P) \cong \frac{R \cap [F,F]}{[F,R]}.$$

Proof.

(i) This follows immediately from Lemma 2.6.4.

(ii) This is a consequence of (i) and Proposition 2.6.3.

(iii) Since $F \wedge F \cong [F,F]$, using the exact sequence (2.6.1), we have

$$P \wedge P \cong \frac{[F,F]}{[F,R]}.$$

Then Lemma 2.6.4 completes the proof. □

Theorem 2.6.6. *Let M be a graded ideal of a Lie superalgebra P. Then there is an exact sequence*

$$\mathsf{Ker}(P \wedge M \to P) \to \mathrm{H}_2(P) \to \mathrm{H}_2(P/M) \to \frac{M}{[P,M]} \to \mathrm{H}_1(P) \to \mathrm{H}_1(P/M) \to 0.$$

Proof. By using the exact sequence (2.6.1) we have the following commutative diagram of Lie superalgebras with exact rows

$$\begin{array}{ccccccccc}
 & & M \wedge P & \longrightarrow & P \wedge P & \longrightarrow & \frac{P}{M} \wedge \frac{P}{M} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & P & \longrightarrow & \frac{P}{M} & \longrightarrow & 0.
\end{array}$$

Since $\mathrm{Coker}(M \wedge P \cong P \wedge M \to M) \cong M/[P,M]$ and $\mathrm{Coker}(P \wedge P \to P) \cong P/[P,P] \cong \mathrm{H}_1(P)$, then the assertion follows by using snake lemma and Corollary 2.6.5(i). □

In particular, if P is a Lie algebra and M is an ideal of P, then this sequence coincides with the six-term exact sequence in the homology of Lie algebras obtained in [7].

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Chapter 3

Universal central extensions of $\mathfrak{sl}(m, n, A)$

Abstract

We find the universal central extension of the matrix superalgebras $\mathfrak{sl}(m, n, A)$ where A is an associative superalgebra and $m + n = 3, 4$ and its relation with the Steinberg superalgebra $\mathfrak{st}(m, n, A)$. We calculate $\mathrm{H}_2\left(\mathfrak{sl}(m, n, A)\right)$ and $\mathrm{H}_2\left(\mathfrak{st}(m, n, A)\right)$. Finally, we introduce a new method using the non-abelian tensor product of Lie superalgebras to find the connection between $\mathrm{H}_2\left(\mathfrak{sl}(m, n, A)\right)$ and the cyclic homology of associative superalgebras for $m + n \geqslant 3$.

Reference

X. García-Martínez and M. Ladra, *Universal central extensions of* $\mathfrak{sl}(m, n, A)$ *of small rank over associative superalgebras*, Turkish J. Math., 2017, doi:10.3906/mat-1604-3.

3.1 Introduction

The study of central extensions plays an important role in the theory of groups or Lie algebras and has numerous applications going through physics, representation theory or homological algebra. They have been studied by many people in the context of Lie algebras as [8, 15], etc. The universal central extension is a key object in this study, since it simplifies the task of finding all central extensions and moreover, its kernel is the second homology group. In [4] the universal central extension of Lie algebras is constructed as a non-abelian tensor product, extended to Lie superalgebras in [7]; and in [9, 13] some of the results of [8] are extended to Lie superalgebras and the universal central extension is constructed. The main problem of these constructions is that they are usually hard to compute.

The concrete problem of finding the universal central extension of $\mathfrak{sl}_n(A)$ for $n \geqslant 5$ was solved in [11]. It is a very important result which involves Steinberg Lie algebras (see [2, 5]) and allowed to develop the additive K-theory. If $n \geqslant 5$, $\mathfrak{st}_n(A)$ is the universal central extension of $\mathfrak{sl}_n(A)$ and if A is K-free, the kernel is isomorphic to the first cyclic homology $\mathrm{HC}_1(A)$. The problem of finding the universal central extension of $\mathfrak{sl}_n(A)$ and $\mathfrak{st}_n(A)$ for $n = 3, 4$ was solved years later in [6]. In [12] the universal central extension of the Lie superalgebras $\mathfrak{sl}(m,n,A)$ and $\mathfrak{st}(m,n,A)$ is computed with $m+n \geqslant 5$, where A is an associative algebra, and the remaining cases where $m+n = 3, 4$ are solved in [14].

If A is an associative superalgebra, the universal central extension of $\mathfrak{sl}_n(A)$ is computed in [3] for all $n \geqslant 3$. The case $\mathfrak{sl}(m,n,A)$ is studied in [7] for $m+n \geqslant 5$, leaving as an open problem the cases $m+n = 3, 4$. In this paper, we will solve these specific cases in order to complete the computation of the universal central extension of $\mathfrak{sl}(m,n,A)$ where A is an associative superalgebra and $m+n \geqslant 3$; and therefore giving a complete characterization of the second homology $\mathrm{H}_2\big(\mathfrak{st}(m,n,A)\big)$ for $m+n \geqslant 3$ (Theorem 3.8.1). Moreover, we introduce a new technique using the non-abelian tensor product of Lie superalgebras defined in [7] to relate $\mathrm{H}_2\big(\mathfrak{st}(m,n,A)\big)$ and the cyclic homology of associative superalgebras for $m+n \geqslant 3$ (Theorem 3.8.2).

The organization of this paper is the following. In Section 3.2 we give some preliminary well-known results and some technical lemmas about $\mathfrak{sl}(m,n,A)$ and $\mathfrak{st}(m,n,A)$. In Section 3.3 we adapt the classical construction of a central extension from a super 2-cocycle in Lie superalgebras. In Section 3.4 we start with the case of $\mathfrak{sl}(2,1,A)$ and we show that its universal central extension is $\mathfrak{st}(2,1,A)$, constructing a (unique) homomorphism to any central extension. In Section 3.5 we find the universal central extension of $\mathfrak{st}(3,1,A)$ (which consequently will be the universal central extension of $\mathfrak{sl}(3,1,A)$) via the construction of a super 2-cocycle; repeating the procedure for $\mathfrak{st}(2,2,A)$ in Section 3.6. In Section 3.7 we relate the second homology of $\mathfrak{sl}(m,n,A)$ with cyclic homology. Finally, in Section 3.8 we give concluding remarks establishing a combination of the results presented here with results of [3, 7] to give the full computation of $\mathrm{H}_2\left(\mathfrak{st}(m,n,A)\right)$ and $\mathrm{H}_2\left(\mathfrak{sl}(m,n,A)\right)$ for $m+n \geqslant 3$.

3.2 The Lie superalgebras $\mathfrak{sl}(m, n, A)$ and $\mathfrak{st}(m, n, A)$

Throughout this paper we consider K as a unital commutative ring and $A = A_{\bar{0}} \oplus A_{\bar{1}}$ an associative unital K-superalgebra. For any $m, n \in \mathbb{Z}_+$, let $\{1, \ldots, m\} \cup \{m+1, \ldots, m+n\}$ be the $\mathbb{Z}_2$-graded set, where the first set is the even part and the second one the odd part. We now consider $\mathrm{Mat}(m, n, A)$ the $(m+n) \times (m+n)$ matrices with coefficients in A. It is defined a $\mathbb{Z}_2$-graduation where homogeneous elements are matrices, denoted by $E_{ij}(a)$, having $a \in A_{\bar{0}}, A_{\bar{1}}$ at position (i, j) and zero elsewhere and $|E_{ij}(a)| = |i| + |j| + |a|$. With this graduation we define the associative superalgebra $\mathfrak{gl}(m, n, A)$ which underlying set is $\mathrm{Mat}(m, n, A)$ with the usual matrix product and it is endowed by a Lie superalgebra structure with the usual bracket $[x, y] = xy - (-1)^{|x||y|}yx$.

Assuming that $m + n \geqslant 3$, we define the *special Lie superalgebra*

$$\mathfrak{sl}(m, n, A) = [\mathfrak{gl}(m, n, A), \mathfrak{gl}(m, n, A)].$$

It is generated by the elements $E_{ij}, 1 \leqslant i \neq j \leqslant m+n$, $a \in A$, with bracket

$$[E_{ij}(a), E_{kl}(b)] = \delta_{jk}E_{il}(ab) - (-1)^{|E_{ij}(a)||E_{kl}(b)|}\delta_{li}E_{kj}(ba).$$

In [1] is introduced a generalization of the supertrace for $x \in \mathfrak{gl}(m, n, A)$, defined as follows:

$$\mathrm{Str}_1(x) = \sum_{i=1}^{m+n} (-1)^{|i|(|i|+|x_{ii}|)} x_{ii},$$

where x_{ii} represents the element of x in the position (i, i). It is straightforward that $\mathfrak{sl}(m, n, A) = \{x \in \mathfrak{gl}(m, n, A) : \mathrm{Str}_1(x) \in [A, A]\}$ and that $\mathfrak{sl}(m, n, A)$ is perfect.

For $m + n \geqslant 3$, the *Steinberg Lie superalgebra* $\mathfrak{st}(m, n, A)$ is defined as the Lie superalgebra over K generated by homogeneous $F_{ij}(a)$, $1 \leqslant i \neq j \leqslant m+n$, and $a \in A$ homogeneous, with grading $|F_{ij}(a)| = |i| + |j| + |a|$, satisfying the following relations:

$$a \mapsto F_{ij}(a) \quad \text{is a } K\text{-linear map}, \tag{3.2.1}$$
$$[F_{ij}(a), F_{jk}(b)] = F_{ik}(ab), \quad \text{for distinct } i,j,k, \tag{3.2.2}$$
$$[F_{ij}(a), F_{kl}(b)] = 0, \quad \text{for } j \neq k, i \neq l, \tag{3.2.3}$$

where $a,b \in A$, $1 \leqslant i,j,k,l \leqslant m+n$. Note that $\mathfrak{st}(m,n,A)$ is a perfect Lie algebra and there is a canonical central extension

$$\varphi\colon \mathfrak{st}(m,n,A) \to \mathfrak{sl}(m,n,A), \qquad \varphi\big(F_{ij}(a)\big) \mapsto E_{ij}(a).$$

Using a completely new technique, in [7] it is shown that if $m+n \geqslant 5$, this epimorphism is the universal central extension of $\mathfrak{sl}(m,n,A)$. The remaining cases, when $m+n=3$ or 4, are left as an open problem and they are the object of study of this paper. Our procedure to solve the problem is to find the universal central extension of $\mathfrak{st}(m,n,A)$ and by [13, Corollary 1.9] it will be the universal central extension of $\mathfrak{sl}(m,n,A)$.

We begin giving some relations in $\mathfrak{st}(m,n,A)$ that will be useful. Let

$$H_{ij}(a,b) = [F_{ij}(a), F_{ji}(b)],$$
$$h(a,b) = H_{1j}(a,b) - (-1)^{|a||b|} H_{1j}(1,ba),$$

for $1 \leqslant i \neq j \leqslant m+n, a \in A$. It is well defined since $h(a,b)$ does not depend on j, for $j \neq 1$. We recall that $|H_{ij}(a,b)| = |a| + |b|$ for homogeneous $a,b \in A$.

Lemma 3.2.1. *We have the following identities in* $\mathfrak{st}(m,n,A)$*,*

$$H_{ij}(a,b) = -(-1)^{(|i|+|j|+|a|)(|i|+|j|+|b|)} H_{ji}(b,a), \tag{3.2.4}$$
$$[H_{ij}(a,b), F_{ik}(c)] = F_{ik}(abc), \tag{3.2.5}$$
$$[H_{ij}(a,b), F_{ki}(c)] = -(-1)^{(|a|+|b|)(|i|+|k|+|c|)} F_{ki}(cab), \tag{3.2.6}$$
$$[H_{ij}(a,b), F_{kj}(c)] = (-1)^{(|i|+|j|+|a|)(|i|+|j|+|b|)+(|a|+|b|)(|j|+|k|+|c|)} F_{kj}(cba) \tag{3.2.7}$$
$$[H_{ij}(a,b), F_{ij}(c)] = F_{ij}\big(abc + (-1)^{(|i|+|j|+|a||b|+|b||c|+|c||a|)} cba\big), \tag{3.2.8}$$
$$[H_{ij}(a,b), F_{kl}(c)] = 0, \tag{3.2.9}$$
$$[h(a,b), F_{1i}(c)] = F_{1i}\big((ab - (-1)^{|a||b|} ba)c\big), \tag{3.2.10}$$
$$[h(a,b), F_{jk}(c)] = 0 \text{ for } j,k \geqslant 2. \tag{3.2.11}$$

for homogeneous $a,b,c \in A$ *and* i,j,k,l *distinct.*

Proof. Relations (3.2.4)–(3.2.9) are just consequences of antisymmetry and Jacobi identities. To check (3.2.10) and (3.2.11) we need to apply (3.2.5) and (3.2.9) to the definition of $h(a,b)$. □

The following lemma gives a better understanding of the structure of $\mathfrak{st}(m,n,A)$.

Lemma 3.2.2. *Let $F_{ij}(A)$ be the subalgebra generated by $F_{ij}(a)$, $\mathcal{N}^+$ the subalgebra generated by $F_{ij}(a)$ for $1 \leqslant i < j \leqslant m+n$, $\mathcal{N}^-$ the subalgebra generated by $F_{ij}(a)$ for $1 \leqslant j < i \leqslant m+n$ and $\mathcal{H}$ the subalgebra generated by $H_{ij}(a,b)$, for all $a,b \in A$. Then*

$$\mathcal{N}^+ = \bigoplus_{1\leqslant i<j\leqslant m+n} F_{ij}(A),$$
$$\mathcal{N}^- = \bigoplus_{1\leqslant j<i\leqslant m+n} F_{ij}(A),$$
$$\mathcal{H} = h(A,A) \oplus \left(\bigoplus_{j=2}^{m+n} H_{1j}(1,A)\right),$$

and we have the decomposition

$$\mathfrak{st}(m,n,A) = \mathcal{N}^+ \oplus \mathcal{H} \oplus \mathcal{N}^- = h(A,A) \oplus \left(\bigoplus_{j=2}^{m+n} H_{1j}(1,A)\right) \bigoplus_{1\leqslant i\neq j\leqslant m+n} F_{ij}(A).$$

□

Definition 3.2.3. Let $\mathcal{I}_m$ be the graded ideal of A generated by the elements ma (i.e. $a + \cdots + a$, m times) and $ab - (-1)^{|a||b|}ba$. Let $A_m = A/\mathcal{I}_m$ be the quotient algebra and denote by $\bar{a} = a + \mathcal{I}_m$ its elements.

Lemma 3.2.4 ([3]). $\mathcal{I}_m = mA + A[A,A]$ *and* $[A,A]A = A[A,A]$. □

3.3 Central extensions of $\mathfrak{sl}(m,n,A)$ and cocycles

Definition 3.3.1. Let L be a Lie superalgebra and $\mathcal{W}$ be a K-free supermodule. A *super 2-cocycle* is a K-bilinear map $\psi\colon L \times L \to \mathcal{W}$ such that

$$\psi(x,y) = -(-1)^{|x||y|}\psi(y,x),$$
$$(-1)^{|x||z|}\psi([x,y],z) + (-1)^{|x||y|}\psi([y,z],x) + (-1)^{|y||z|}\psi([z,x],y) = 0,$$
$$\psi(x_{\bar 0}, x_{\bar 0}) = 0,$$

for all $x,y,z \in L$, $x_{\bar 0} \in L_{\bar 0}$.

Given an even super 2-cocycle ψ, we can construct a central extension ([13]) $L \oplus \mathcal{W} \to L$, $(x,w) \mapsto x$, where the bracket is given by $[(x,w_1),(y,w_2)] = \big([x,y],\psi(x,y)\big)$ (see [13]). In the particular case of $L = \mathfrak{st}(m,n,A)$ and the super 2-cocycle being surjective, this construction can be described in a different way using generators and relations.

Definition 3.3.2. Let $\psi\colon \mathfrak{st}(m,n,A) \times \mathfrak{st}(m,n,A) \to \mathcal{W}$ be an even super 2-cocycle, i.e. a super 2-cocycle such that $|\psi(x,y)| = |x| + |y|$ for homogeneous $x,y \in \mathfrak{st}(m,n,A)$. Let $\mathfrak{st}(m,n,A)^\sharp$ be the Lie superalgebra generated by the elements $F_{ij}(a)^\sharp$ with homogeneous $a \in A$, $1 \leqslant i \neq j \leqslant m+n$, with degree $|F^\sharp_{ij}(a)| = |i| + |j| + |a|$ and by the elements of $\mathcal{W}$, with the relations

$$\begin{aligned} a &\mapsto F^\sharp_{ij}(a) \quad \text{is a } K\text{-linear map},\\ [\mathcal{W},\mathcal{W}] &= [F^\sharp_{ij}(a),\mathcal{W}] = 0,\\ [F^\sharp_{ij}(a),F^\sharp_{jk}(b)] &= F^\sharp_{ik}(ab) + \psi\big(F_{ij}(a),F_{jk}(b)\big) \quad \text{for distinct i, j, k},\\ [F^\sharp_{ij}(a),F^\sharp_{kl}(b)] &= \psi\big(F_{ij}(a),F_{kl}(b)\big) \text{ for } i \neq j \neq k \neq l \neq i, \end{aligned}$$

where $a,b \in A$.

Lemma 3.3.3. *If $\mathfrak{st}(m,n,A)' = \mathfrak{st}(m,n,A) \oplus \mathcal{W}$ is a central extension constructed from a surjective super 2-cocycle $\psi\colon \mathfrak{st}(m,n,A) \times \mathfrak{st}(m,n,A) \to \mathcal{W}$ then there is an isomorphism $\rho\colon \mathfrak{st}(m,n,A)^\sharp \to \mathfrak{st}(m,n,A)'$ where $\rho\big(F^\sharp_{ij}(a)\big) = F_{ij}$ and $\rho(w) = w$.*

Proof. The proof of [3, Lemma 1] can be easily adapted. □

As before, we denote $H^\sharp_{ij}(a,b) = [F^\sharp_{ij}(a),F^\sharp_{ji}(b)]$ and $h^\sharp(a,b) = H^\sharp_{1j}(a,b) - (-1)^{|a||b|}H^\sharp_{1j}(1,ba)$. Therefore, $h^\sharp$ is independent of j and we have the analogue decomposition lemma.

Lemma 3.3.4. *We can decompose the Lie superalgebra $\mathfrak{st}(m,n,A)^\sharp$ generated by a surjective super 2-cocycle $\psi\colon \mathfrak{st}(m,n,A)\times\mathfrak{st}(m,n,A)\to\mathcal{W}$ in the following way:*

$$\mathfrak{st}(m,n,A)^\sharp = \mathcal{W} \oplus h^\sharp(A,A) \oplus \left(\bigoplus_{j=2}^{m+n} H^\sharp_{1j}(1,A)\right) \bigoplus_{1\leqslant i\neq j\leqslant m+n} F^\sharp_{ij}(A).$$

□

3.4 Universal central extension of $\mathfrak{st}(2,1,A)$

In this section we study the case when $m+n=3$ and prove that $\mathfrak{st}(2,1,A)$ is the universal central extension of $\mathfrak{sl}(2,1,A)$.

Theorem 3.4.1. *If $\tau\colon \widetilde{\mathfrak{st}}(2,1,A)\to\mathfrak{st}(2,1,A)$ is a central extension, then there exists a unique section $\eta\colon \mathfrak{st}(2,1,A)\to\widetilde{\mathfrak{st}}(2,1,A)$.*

Proof. We will directly obtain a Lie superalgebra homomorphism $\eta\colon \mathfrak{st}(2,1,A)\to\widetilde{\mathfrak{st}}(2,1,A)$, such that $\tau\circ\eta=\mathrm{id}$ and since $\mathfrak{st}(2,1,A)$ is perfect it must be unique. Let

$$0\longrightarrow \mathrm{V}\longrightarrow \widetilde{\mathfrak{st}}(2,1,A)\xrightarrow{\ \tau\ }\mathfrak{st}(2,1,A)\longrightarrow 0$$

be a central extension. We choose a preimage for $F_{ij}(a)$ denoted by $\widetilde{F}_{ij}(a)$ and extend it by K-linearity to all $a\in A$.

We define $\widetilde{H}_{ij}(a,b)=[\widetilde{F}_{ij}(a),\widetilde{F}_{ji}(b)]$, since it is independent of the choice of $\widetilde{F}_{ij}(a)$. By identity (3.2.4) we know that $[\widetilde{H}_{ik}(1,1),\widetilde{F}_{ij}(a)]=\widetilde{F}_{ij}(a)+v_{ij}(a)$, where $v_{ij}(a)\in\mathrm{V}$, so we will replace $\widetilde{F}_{ij}(a)$ by $\widetilde{F}_{ij}(a)+v_{ij}(a)$.

It suffices to show that these $\widetilde{F}_{ij}(a)$ satisfy relations (3.2.1)–(3.2.3) because our K-linear section $\eta\colon \mathfrak{st}(2,1,A)\to\widetilde{\mathfrak{st}}(2,1,A)$, $F_{ij}(a)\mapsto\widetilde{F}_{ij}(a)$ will be a Lie superalgebra homomorphism and the result is proved. The first relation is immediate by definition.

To see the second one, we use Jacobi identity and the fact that V is in the centre of $\widetilde{\mathfrak{st}}(2,1,A)$.

$$\begin{aligned}\widetilde{F}_{ij}(ab) &= [\widetilde{H}_{ik}(1,1),\widetilde{F}_{ij}(ab)] = \big[\widetilde{H}_{ik}(1,1),[\widetilde{F}_{ik}(a),\widetilde{F}_{kj}(b)]\big]\\ &= \big[[\widetilde{H}_{ik}(1,1),\widetilde{F}_{ik}(a)],\widetilde{F}_{kj}(b)\big] + \big[\widetilde{F}_{ik}(a),[\widetilde{H}_{ik}(1,1),\widetilde{F}_{kj}(b)]\big]\\ &= [\widetilde{F}_{ik}(a+(-1)^{|i|+|k|}a),\widetilde{F}_{kj}(b)] + [\widetilde{F}_{ik}(a),-(-1)^{(|i|+|k|)(|i|+|k|)}\widetilde{F}_{kj}(b)]\\ &= [\widetilde{F}_{ik}(a),\widetilde{F}_{kj}(b)].\end{aligned}$$

Now we check that the remaining brackets vanish.

$$\begin{aligned}[\widetilde{F}_{ij}(a),\widetilde{F}_{ij}(b)] &= \big[\widetilde{F}_{ij}(a),[\widetilde{F}_{ik}(b),\widetilde{F}_{kj}(1)]\big]\\ &= \big[[\widetilde{F}_{ij}(a),\widetilde{F}_{ik}(b)],\widetilde{F}_{kj}(1)\big]\\ &\quad+(-1)^{(|i|+|j|+|a|)(|i|+|k|+|b|)}\big[\widetilde{F}_{ik}(b),[\widetilde{F}_{ij}(a),\widetilde{F}_{kj}(1)]\big]=0.\end{aligned}$$

To see that $[\widetilde{F}_{ij}(a),\widetilde{F}_{ik}(b)]=0$ we can assume $|i|+|j|=\bar{1}$, then

$$\begin{aligned}0 &= (-1)^{\bar{1}+|a|}\big[\widetilde{H}_{ij}(1,1),[\widetilde{F}_{ij}(a),\widetilde{F}_{ik}(b)]\big]\\ &= (-1)^{\bar{1}+|a|}\big[[\widetilde{H}_{ij}(1,1),\widetilde{F}_{ij}(a)],\widetilde{F}_{ik}(b)\big]\\ &\quad+(-1)^{(\bar{1}+|a|)+(|i|+|j|)(|i|+|j|+|a|)}\big[\widetilde{F}_{ij}(a),[\widetilde{H}_{ij}(1,1),\widetilde{F}_{ik}(b)]\big]\\ &= (-1)^{\bar{1}+|a|}[\widetilde{F}_{ij}(a+(-1)^{\bar{1}}a),\widetilde{F}_{ik}(b)]+[\widetilde{F}_{ij}(a),\widetilde{F}_{ik}(b)]=[\widetilde{F}_{ij}(a),\widetilde{F}_{ik}(b)].\end{aligned}$$

If $|i|+|j|=\bar{0}$, we have that $|i|+|k|=\bar{1}$ and the calculation is the same. Therefore, $[\widetilde{F}_{ij}(a),\widetilde{F}_{kl}(b)]=0$ if $j\neq k$ and $i\neq l$, satisfying relation (3.2.3) and completing the proof. □

Corollary 3.4.2. *The universal central extension of* $\mathfrak{sl}(2,1,A)$ *and* $\mathfrak{st}(2,1,A)$ *is* $\mathfrak{st}(2,1,A)$. *Moreover,* $\mathrm{H}_2\big(\mathfrak{st}(2,1,A)\big)=0$.

3.5 Universal central extension of $\mathfrak{st}(3,1,A)$

In this section we find the universal central extension of $\mathfrak{sl}(3,1,A)$. Let S_4 be the symmetric group of degree 4, i.e. the set of all quadruples (i,j,k,l) where $1\leqslant i,j,k,l\leqslant 4$ distinct. We quotient S_4 by Klein's subgroup, formed by $\{(1,2,3,4),(3,2,1,4),(1,4,3,2),(3,4,1,2)\}$, obtaining 6 cosets denoted by P_m. We have a map θ that sends $(i,j,k,l)\mapsto\theta\big((i,j,k,l)\big)=m$ when $(i,j,k,l)\in P_m$.

Let $\Pi(A_2)$ be the K-supermodule A_2 (see Definition 3.2.3) with the parity changed, i.e., $\big(\Pi(A_2)\big)_{\bar{0}}=(A_2)_{\bar{1}}$ and $\big(\Pi(A_2)\big)_{\bar{1}}=(A_2)_{\bar{0}}$. Let $\mathcal{W}=\Pi(A_2)^6$ be the K-supermodule formed by the direct sum of six copies of $\Pi(A_2)$ and consider the maps $\epsilon_m\colon\Pi(A_2)\to\mathcal{W}$, $\epsilon_m(\bar{a})\mapsto(0,\dots,\bar{a},\dots,0)$, in the position m.

Using the decomposition of Lemma 3.2.2 we consider the K-bilinear map

$$\psi\colon\mathfrak{st}(3,1,A)\times\mathfrak{st}(3,1,A)\to\mathcal{W},$$

where

$$\psi\big(F_{ij}(a), F_{kl}(b)\big) = \epsilon_{\theta\left((i,j,k,l)\right)}(\overline{ab}),$$
$$\psi(x,y) = 0 \text{ if } x \text{ or } y \text{ belong to } \mathcal{H}.$$

Lemma 3.5.1. *The K-bilinear map ψ is a super 2-cocycle.*

Proof. Since the grading in $\mathcal{W}$ is changed and exactly one index is odd, we have that

$$|\psi\big(F_{ij}(a), F_{kl}(b)\big)| = |i|+|j|+|a|+|k|+|l|+|b| = |a|+|b|+\bar{1} = |\epsilon_{\theta\left((i,j,k,l)\right)}(\overline{ab})|,$$

for homogeneous $a, b \in A$, so ψ is even.

To complete the proof we can just follow the steps of [6, Lemma 2.2] since $\bar{a} = -\bar{a}$ so signs do not play any important role. □

By the previous lemma, we have a central extension

$$0 \longrightarrow \mathcal{W} \longrightarrow \mathfrak{st}(3,1,A)^\sharp \xrightarrow{\pi} \mathfrak{st}(3,1,A) \longrightarrow 0,$$

where $\mathfrak{st}(3,1,A)^\sharp = \mathfrak{st}(3,1,A) \oplus \mathcal{W}$ is the Lie superalgebra constructed by the surjective super 2-cocycle ψ, defined by the following relations

$$a \mapsto F^\sharp_{ij}(a) \text{ is a } K\text{-linear map}, \tag{3.5.1}$$
$$[\mathcal{W}, \mathcal{W}] = [F^\sharp_{ij}(a), \mathcal{W}] = 0, \tag{3.5.2}$$
$$[F^\sharp_{ij}(a), F^\sharp_{jk}(b)] = F^\sharp_{ik}(ab) \text{ for distinct i, j, k}, \tag{3.5.3}$$
$$[F^\sharp_{ij}(a), F^\sharp_{ij}(a)] = 0, \tag{3.5.4}$$
$$[F^\sharp_{ij}(a), F^\sharp_{ik}(b)] = 0, \tag{3.5.5}$$
$$[F^\sharp_{ij}(a), F^\sharp_{kl}(b)] = \epsilon_{\theta\left((i,j,k,l)\right)}(\overline{ab}) \text{ for distinct } i,j,k,l. \tag{3.5.6}$$

Theorem 3.5.2. *The central extension $0 \to \mathcal{W} \to \mathfrak{st}(3,1,A)^\sharp \to \mathfrak{st}(3,1,A)$ is universal.*

Proof. Let

$$0 \longrightarrow \mathrm{V} \longrightarrow \widetilde{\mathfrak{st}}(3,1,A) \xrightarrow{\tau} \mathfrak{st}(3,1,A) \longrightarrow 0$$

be a central extension. We need to show that there exists a Lie superalgebra homomorphism $\rho\colon \mathfrak{st}(3,1,A)^\sharp \to \widetilde{\mathfrak{st}}(3,1,A)$ such that $\tau\circ\rho=\pi$.

We choose a preimage $\widetilde{F}_{ij}(a)$ of $F_{ij}(a)$ K-linearly for all $a\in A$. Since $\mathrm{V}\subset \mathsf{Z}(\widetilde{\mathfrak{st}}(3,1,A))$, we have that

$$[\widetilde{F}_{ik}(a),\widetilde{F}_{kj}(b)]=\widetilde{F}_{ij}(ab)+v_{ijk}(a,b),$$

for distinct i,j,k, where $v_{ijk}(a,b)\in\mathrm{V}$. Using Jacobi identity we have

$$\begin{aligned}[\widetilde{F}_{ik}(a),\widetilde{F}_{kj}(cb)] &= \big[\widetilde{F}_{ik}(a),[\widetilde{F}_{kl}(c),\widetilde{F}_{lj}(b)]\big]\\ &= \big[[\widetilde{F}_{ik}(a),\widetilde{F}_{kl}(c)],\widetilde{F}_{lj}(b)\big]\\ &\quad+(-1)^{(|i|+|k|+|a|)(|k|+|l|+|c|)}\big[\widetilde{F}_{kl}(c),[\widetilde{F}_{ik}(a),\widetilde{F}_{lj}(b)]\big]\\ &= [\widetilde{F}_{il}(ac),\widetilde{F}_{lj}(b)],\end{aligned}$$

so choosing $c=1$ we have the identities $v_{ijk}(a,b)=v_{ijl}(a,b)$ and $[\widetilde{F}_{ik}(a),\widetilde{F}_{kj}(b)]=[\widetilde{F}_{il}(a),\widetilde{F}_{lj}(b)]$. This means that $v_{ijk}(a,b)$ is independent of the choice of k so we have

$$[\widetilde{F}_{ik}(a),\widetilde{F}_{kj}(b)]=\widetilde{F}_{ij}(ab)+v_{ij}(a,b),$$

and

$$[\widetilde{F}_{ik}(1),\widetilde{F}_{kj}(b)]=\widetilde{F}_{ij}(b)+v_{ij}(1,b).$$

Therefore, we can replace $\widetilde{F}_{ij}(b)$ by $\widetilde{F}_{ij}(b)+v_{ij}(1,b)$. We want to define $\rho\big(F^\sharp_{ij}(a)\big)=\widetilde{F}_{ij}(a)$ so will see that these elements satisfy relations (3.5.1)–(3.5.6).

Relations (3.5.1), (3.5.2) and (3.5.3) are straightforward by definition. To see relation (3.5.4), we choose i,j,k distinct

$$\begin{aligned}[\widetilde{F}_{ij}(a),\widetilde{F}_{ij}(b)] &= \big[\widetilde{F}_{ij}(a),[\widetilde{F}_{ik}(b),\widetilde{F}_{kj}(1)]\big]\\ &= \big[[\widetilde{F}_{ij}(a),\widetilde{F}_{ik}(b)],\widetilde{F}_{kj}(1)\big]\\ &\quad+(-1)^{(|i|+|j|+|a|)(|i|+|k|+|b|)}\big[\widetilde{F}_{ik}(b),[\widetilde{F}_{ij}(a),\widetilde{F}_{kj}(1)]\big]\\ &= 0.\end{aligned}$$

For relation (3.5.5), taking i,j,k,l distinct, we have

$$\begin{aligned}[\widetilde{F}_{ij}(a),\widetilde{F}_{ik}(b)] &= \big[\widetilde{F}_{ij}(a),[\widetilde{F}_{il}(b),\widetilde{F}_{ik}(1)]\big]\\ &= \big[[\widetilde{F}_{ij}(a),\widetilde{F}_{il}(b)],\widetilde{F}_{ik}(1)\big]\\ &\quad + (-1)^{(|i|+|j|+|a|)(|i|+|l|+|b|)}\big[\widetilde{F}_{il}(b),[\widetilde{F}_{ij}(a),\widetilde{F}_{ik}(1)]\big]\\ &= 0.\end{aligned}$$

To check relation (3.5.6) we define $\widetilde{H}_{ij}(a,b) = [\widetilde{F}_{ij}(a),\widetilde{F}_{ji}(b)]$ and following the steps of Lemma 3.2.1 we can check that for distinct i,j,k,l,

$$\begin{aligned}\widetilde{H}_{ij}(a,b) &= -(-1)^{(|i|+|j|+|a|)(|i|+|j|+|b|)}\widetilde{H}_{ji}(b,a),\\ [\widetilde{H}_{ij}(a,b),\widetilde{F}_{ik}(c)] &= \widetilde{F}_{ik}(abc),\\ [\widetilde{H}_{ij}(a,b),\widetilde{F}_{ki}(c)] &= -(-1)^{(|a|+|b|)(|i|+|k|+|c|)}\widetilde{F}_{ki}(cab),\\ [\widetilde{H}_{ij}(a,b),\widetilde{F}_{kj}(c)] &= (-1)^{(|i|+|j|+|a|)(|i|+|j|+|b|)+(|a|+|b|)(|j|+|k|+|c|)}\widetilde{F}_{kj}(cba),\\ [\widetilde{H}_{ij}(a,b),\widetilde{F}_{ij}(c)] &= \widetilde{F}_{ij}(abc+(-1)^{(|i|+|j|+|a||b|+|b||c|+|c||a|)}cba),\\ [\widetilde{H}_{ij}(a,b),\widetilde{F}_{kl}(c)] &= 0.\end{aligned}$$

When i,j,k,l are distinct we denote

$$[\widetilde{F}_{ij}(a),\widetilde{F}_{kl}(1)] = v_{ijkl}(a),$$

where $v_{ijkl}(a)\in \mathrm{V}$. We want that $\rho(\epsilon_{\theta\left((i,j,k,l)\right)}(\overline{ab})) = v_{ijkl}(ab)$, since

$$\begin{aligned}[\widetilde{F}_{ij}(a),\widetilde{F}_{kl}(b)] &= [\rho\big(F^{\sharp}_{ij}(a)\big),\rho\big(F^{\sharp}_{kl}(b)\big)]\\ &= \rho\big([F^{\sharp}_{ij}(a),F^{\sharp}_{kl}(b)]\big) = \rho(\epsilon_{\theta\left((i,j,k,l)\right)}(\overline{ab})) = v_{ijkl}(ab).\end{aligned}$$

Thus, we have to check that

(R1) $2v_{ijkl}(a) = 0$,

(R2) $v_{ijkl}(a) = v_{kjil}(a) = v_{ilkj}(a) = v_{klij}(a)$,

(R3) $v_{ijkl}(a[b,c]) = 0$,

(R4) $[\widetilde{F}_{ij}(a),\widetilde{F}_{kl}(b)] = v_{ijkl}(ab)$.

Assume $|i|+|j|=\bar{0}$,

$$\begin{aligned}0 &= \big[\widetilde{H}_{ij}(a,b),[\widetilde{F}_{ij}(c),\widetilde{F}_{kl}(1)]\big]\\ &= \big[[\widetilde{H}_{ij}(a,b),\widetilde{F}_{ij}(c)],\widetilde{F}_{kl}(1)\big]-\big[\widetilde{F}_{ij}(a),[\widetilde{H}_{ij}(1,1),\widetilde{F}_{kl}(1)]\big]\\ &= [\widetilde{F}_{ij}(abc+(-1)^{|i|+|j|+|a||b|+|b||c|+|c||a|}cba),\widetilde{F}_{kl}(1)]\\ &= v_{ijkl}(abc+(-1)^{|a||b|+|b||c|+|c||a|}cba).\end{aligned}$$

If $b=c=1$, we have that

$$v_{ijkl}(2a)=2v_{ijkl}(a)=0,$$

proving (R1).

If $c=1$, we have that

$$v_{ijkl}(ab-(-1)^{|a||b|}ba)=0,$$

so

$$0=v_{ijkl}(abc+(-1)^{|a||b|+|b||c|+|c||a|}cba)=v_{ijkl}\big((ab+(-1)^{|a||b|}ba)c\big),$$

implying (R2). If $|k|+|l|=\bar{0}$, the calculation is the same.

On the other hand,

$$\begin{aligned}[\widetilde{F}_{ij}(a),\widetilde{F}_{kl}(b)] &= \big[[\widetilde{F}_{ik}(a),\widetilde{F}_{kj}(1)],\widetilde{F}_{kl}(b)\big]\\ &= \big[\widetilde{F}_{ik}(a),[\widetilde{F}_{kj}(1),\widetilde{F}_{kl}(b)]\big]\\ &\quad -(-1)^{(|i|+|k|+|a|)(|k|+|j|)}\big[\widetilde{F}_{kj}(1),[\widetilde{F}_{ik}(a),\widetilde{F}_{kl}(b)]\big]\\ &= (-1)^{(|l|+|k|+|b|)(|k|+|j|)}[\widetilde{F}_{il}(ab),\widetilde{F}_{kj}(1)]\\ &= v_{ilkj}(ab),\end{aligned}$$

since the sign does not play any role. Choosing $b=1$ and using (R1), we have that

$$v_{ijkl}(a)=v_{ilkj}(a).$$

Doing the same but changing the indexes we have relations (R3) and (R4).

Thus, the morphism $\rho\colon \mathfrak{st}(3,1,A)^\sharp\to\widetilde{\mathfrak{st}}(3,1,A)$ defined by

$$\rho\big(F^\sharp_{ij}(a)\big)=\widetilde{F}_{ij}(a)\qquad\text{and}\qquad\rho(\epsilon_{\theta\left((i,j,k,l)\right)}(\overline{ab}))=v_{ijkl}(ab)$$

is actually a Lie superalgebra homomorphism completing the proof. □

Corollary 3.5.3. *The universal central extension of* $\mathfrak{sl}(3,1,A)$ *is* $\mathfrak{st}(3,1,A)^\sharp\cong\mathfrak{sl}(3,1,A)\oplus\Pi(A_2)^6$. *Moreover,* $\mathrm{H}_2\left(\mathfrak{st}(3,1,A)\right)\cong\mathcal{W}\cong\Pi(A_2)^6$.

3.6 Universal central extension of $\mathfrak{st}(2,2,A)$

In this section we find the universal central extension of $\mathfrak{st}(2,2,A)$. As in the previous section we consider the partition of S_4 but with a small difference. Not all the cosets will be considered as equals. The coset formed by

$$\{(1,3,2,4),(1,4,2,3),(2,3,1,4),(2,4,1,3)\},$$

is named P_5 the one formed by

$$\{(3,1,4,2),(3,2,4,1),(4,1,3,2),(4,2,3,1)\},$$

is named P_6. The order of the other cosets $P_1,\ldots,P_4$, will not be relevant. Note that all the elements of P_5 and P_6 have the property that $|i|=|k|$, $|j|=|l|$ and $|i|+|j|=|k|+|l|=\bar{1}$.

Let $\sigma\colon S_4\to\{-1,1\}$ be a map defined by

$$\sigma\big((i,j,k,l)\big)=1 \text{ if } (i,j,k,l)\in P_1,P_2,P_3 \text{ or } P_4,$$

in P_5,

$$\begin{aligned}
\sigma\big((i,j,k,l)\big)&=1 && \text{if } (i,j,k,l)=(1,3,2,4) \text{ or } (2,4,1,3),\\
\sigma\big((i,j,k,l)\big)&=-1 && \text{if } (i,j,k,l)=(1,4,2,3) \text{ or } (2,3,1,4),
\end{aligned}$$

and in P_6,

$$\begin{aligned}
\sigma\big((i,j,k,l)\big)&=1 && \text{if } (i,j,k,l)=(3,1,4,2) \text{ or } (4,2,3,1),\\
\sigma\big((i,j,k,l)\big)&=-1 && \text{if } (i,j,k,l)=(3,2,4,1) \text{ or } (4,1,3,2).
\end{aligned}$$

Furthermore, let $\mathcal{W}=A_2^4\oplus A_0^2$ be K-supermodule formed by the direct sum of four copies of A_2 and two copies of A_0 and the maps $\epsilon_m(\bar{a})=(0,\ldots,\bar{a},\ldots,0)$ in position m.

Using the decomposition of Lemma 3.2.2 we consider the K-bilinear map

$$\psi\colon \mathfrak{st}(2,2,A)\times\mathfrak{st}(2,2,A)\to\mathcal{W},$$

where

$$\begin{aligned}
&\psi\big(F_{ij}(a),F_{kl}(b)\big)=\epsilon_{\theta\left((i,j,k,l)\right)}(\overline{ab}), && \text{if } (i,j,k,l)\in P_1,P_2,P_3,P_4\\
&\psi\big(F_{ij}(a),F_{kl}(b)\big)=(-1)^{|b|}\sigma\big((i,j,k,l)\big)\epsilon_{\theta\left((i,j,k,l)\right)}(\overline{ab}), && \text{if } (i,j,k,l)\in P_5 \text{ or } P_6,\\
&\psi(x,y)=0 && \text{if } x \text{ or } y \text{ belong to } \mathcal{H}.
\end{aligned}$$

Lemma 3.6.1. *The K-bilinear map ψ is a super 2-cocycle.*

Proof. The map is even since $|i|+|j|+|k|+|l|=\bar{0}$. To check antisymmetry, it suffices to see what happens when $(i,j,k,l)\in P_5$ or P_6 since in the other cases the signs do not make any difference since A_2 and A_0 are commutative. Let $(i,j,k,l)\in P_5$, we know that $|i|+|j|=|k|+|l|=\bar{1}$,

$$\begin{aligned}
-(-1)^{|F_{ij}(a)||F_{kl}(b)|}\psi\big(F_{kl}(b),F_{ij}(a)\big) &= -(-1)^{(|i|+|j|+|a|)(|k|+|l|+|b|)}\psi\big(F_{kl}(b),F_{ij}(a)\big)\\
&= -(-1)^{(\bar{1}+|a|)(\bar{1}+|b|)}(-1)^{|a|}\sigma\big((k,l,i,j)\big)\epsilon_5(\overline{ba})\\
&= (-1)^{|b|+|a||b|}\sigma\big((k,l,i,j)\big)\epsilon_5((-1)^{|a||b|}\overline{ab})\\
&= (-1)^{|b|}\sigma\big((i,j,k,l)\big)\epsilon_{\theta\big((i,j,k,l)\big)}(\overline{ab})\\
&= \psi\big(F_{ij}(a),F_{kl}(b)\big),
\end{aligned}$$

since $\sigma\big((i,j,k,l)\big)=\sigma\big((k,l,i,j)\big)$ and $\overline{ab}=(-1)^{|a||b|}\overline{ba}$. If (i,j,k,l) belongs to P_6 it is analogue.

The identity $\psi(x_{\bar{0}},x_{\bar{0}})=0$ where $x_{\bar{0}}\in\big(\mathfrak{st}(2,2,A)\big)_{\bar{0}}$ is straightforward by definition. The last step is to check Jacobi identity. In order to ease notation, we denote by $J(x,y,z)$ the expression

$$(-1)^{|x||z|}\psi([x,y],z)+(-1)^{|x||y|}\psi([y,z],x)+(-1)^{|y||z|}\psi([z,x],y).$$

We have to check that $J(x,y,z)=0$ for all $x,y,z\in\mathfrak{st}(2,2,A)$.

Let $\psi([x,y],z)\neq 0$. Using the decomposition of Lemma 3.2.2 we see that at most one of x,y belongs to $\mathcal{H}$. We can assume that $x\in\mathcal{H}$. To exclude trivial cases we need that $y=F_{ij}(a)$ and $z=F_{kl}(b)$, where i,j,k,l are distinct. If $(i,j,k,l)\in P_1,\dots,P_4$, the signs does not make any difference so the proof is the same as in [6, Lemma 2.2]. Therefore, we just need to check when $(i,j,k,l)=(1,3,2,4)\in P_5$ since the other cases are similar.

If $x=h(c,d)$, then

$$\begin{aligned}
J(x,y,z) &= (-1)^{(|c|+|d|)(|b|+\bar{1})}\psi\big([h(c,d),F_{13}(a)],F_{24}(b)\big)\\
&\quad+(-1)^{(|a|+\bar{1})(|b|+\bar{1})}\psi\big([F_{24},h(c,d)],F_{13}(b)\big)\\
&= (-1)^{(|c|+|d|)(|b|+\bar{1})}\psi\big(F_{13}\big((ab-(-1)^{|a||b|}ba)c\big),F_{24}(b)\big)+0\\
&= (-1)^{(|c|+|d|)(|b|+\bar{1})+|b|}\sigma\big((1,3,2,4)\big)\epsilon_5\big(\overline{(ab-(-1)^{|a||b|}ba)cb}\big)\\
&= 0.
\end{aligned}$$

If $x = H_{12}(1,c)$, then

$$\begin{aligned}
J(x,y,z) &= (-1)^{|c|(|a|+\bar{1})}\psi\big([H_{12}(1,c), F_{13}(a)], F_{24}(b)\big) \\
&\quad + (-1)^{(|a|+\bar{1})(|b|+\bar{1})}\psi\big([F_{24}(b), H_{12}(1,c)], F_{13}(a)\big) \\
&= (-1)^{|c|(|a|+\bar{1})}\psi\big(F_{13}(ca), F_{24}(b)\big) \\
&\quad + (-1)^{(|a|+\bar{1})(|b|+\bar{1})+|c|(|b|+\bar{1})}\psi\big(F_{24}(cb), F_{13}(a)\big) \\
&= (-1)^{|c|(|b|+\bar{1})+|a|}\sigma\big((1,3,2,4)\big)\epsilon_5(\overline{cab}) \\
&\quad + (-1)^{(|a|+|c|+\bar{1})(|b|+\bar{1})+|b|}\sigma\big((2,4,1,3)\big)\epsilon_5(\overline{cba}) \\
&= (-1)^{|c|(|b|+\bar{1})}\big((-1)^{|a|}\epsilon_5(\overline{cab}) + (-1)^{(|a|+\bar{1})(|b|+\bar{1})+|b|}\epsilon_5(\overline{cba})\big) \\
&= (-1)^{|c|(|b|+\bar{1})+|a|}\big(\epsilon_5(\overline{cab - (-1)^{|a||b|}cba})\big) \\
&= 0.
\end{aligned}$$

If $x = H_{13}(1,c)$, then

$$\begin{aligned}
J(x,y,z) &= (-1)^{|c|(|a|+\bar{1})}\psi\big([H_{13}(1,c), F_{13}(a)], F_{24}(b)\big) \\
&= \psi\big(F_{13}(ca + (-1)^{\bar{1}+|a||c|}ac), F_{24}(b)\big) \\
&= (-1)^{|b|}\sigma\big((1,3,2,4)\big)\epsilon_5(\overline{(ca - (-1)^{|a||c|}ac)b}) \\
&= 0.
\end{aligned}$$

If $x = H_{14}(1,c)$, then

$$\begin{aligned}
J(x,y,z) &= (-1)^{|c|(|b|+\bar{1})}\psi\big([H_{14}(1,c), F_{13}(a)], F_{24}(b)\big) \\
&\quad + (-1)^{(|a|+\bar{1})(|b|+\bar{1})}\psi\big([F_{24}(b), H_{14}(1,c)], F_{13}(a)\big) \\
&= (-1)^{|c|(|b|+\bar{1})}\psi\big(F_{13}(ca), F_{24}(b)\big) \\
&\quad + (-1)^{(|a|+\bar{1})(|b|+\bar{1})+|c|}\psi\big(F_{24}(bc), F_{13}(a)\big) \\
&= (-1)^{|c|(|b|+\bar{1})+|b|}\sigma\big((1,3,2,4)\big)\epsilon_5(\overline{cab}) \\
&\quad + (-1)^{(|a|+\bar{1})(|b|+\bar{1})+|c|+|a|}\sigma\big((2,4,1,3)\big)\epsilon_5(\overline{bca}) \\
&= (-1)^{|b|+|c|}\big((-1)^{|c||b|}\epsilon_5(\overline{cab}) - (-1)^{|a||b|}\epsilon_5(\overline{bca})\big) \\
&= (-1)^{|b|+|c|}\big((-1)^{|c||b|+|b||c|+|a||b|}\epsilon_5(\overline{bca}) - (-1)^{|a||b|}\epsilon_5(\overline{bca})\big) \\
&= 0.
\end{aligned}$$

Assume now that neither $x,y,z \in \mathcal{H}$. If $\psi([x,y],z) \neq 0$ we must have $\psi\big([F_{ik}(a),F_{kj}(b)],F_{kl}(c)\big)$ or $\psi\big([F_{il}(a),F_{lj}(b)],F_{kl}(c)\big)$. Again, if $(i,j,k,l) \in P_1,\dots,P_4$, the sign does not matter so the proof is the same as in [6]. Assume that $(i,j,k,l) = (1,3,2,4) \in P_5$.

If $x = F_{12}(a)$, $y = F_{23}(b)$ and $z = F_{24}(c)$, then

$$\begin{aligned}
J(x,y,z) &= (-1)^{|a|(|c|+\bar{1})}\psi\big(F_{13}(ab),F_{24}(c)\big)\\
&\quad - (-1)^{(|b|+\bar{1})(|c|+\bar{1})+|a|(|c|+\bar{1})}\psi\big(F_{14}(ac),F_{23}(b)\big)\\
&= (-1)^{|a|(|c|+\bar{1})+|c|}\sigma\big((1,3,2,4)\big)\epsilon_5(\overline{abc})\\
&\quad - (-1)^{(|a|+|b|+\bar{1})(|c|+\bar{1})+|b|}\sigma\big((1,4,2,3)\big)\epsilon_5(\overline{acb})\\
&= (-1)^{|a|(|c|+\bar{1})+|c|}\big(\epsilon_5(\overline{abc}) + (-1)^{|b||c|+\bar{1}}\epsilon_5(\overline{acb})\big)\\
&= (-1)^{|a|(|c|+\bar{1})+|c|}\epsilon_5\big(\overline{a(bc-(-1)^{|b||c|}cb)}\big)\\
&= 0.
\end{aligned}$$

If $x = F_{14}(a)$, $y = F_{43}(b)$ and $z = F_{24}(c)$, then

$$\begin{aligned}
J(x,y,z) &= (-1)^{(|a|+\bar{1})(|c|+\bar{1})}\psi\big(F_{13}(ab),F_{24}(c)\big)\\
&\quad - (-1)^{|b|(|a|+\bar{1})+|b|(|c|+\bar{1})}\psi\big(F_{23}(ac),F_{14}(b)\big)\\
&= (-1)^{(|a|+\bar{1})(|c|+\bar{1})+|c|}\sigma\big((1,3,2,4)\big)\epsilon_5(\overline{abc})\\
&\quad - (-1)^{|b|(|a|+|c|)+|a|}\sigma\big((2,3,1,4)\big)\epsilon_5(\overline{cba})\\
&= -(-1)^{|a|}\epsilon_5\big(\overline{(-1)^{|a||c|}abc-(-1)^{|a||b|+|b||c|}cba}\big)\\
&= -(-1)^{|a|}\epsilon_5\big(\overline{(-1)^{|a||c|}abc-(-1)^{|a||b|+|b||c|+|c|(|b|+|a|)}bac}\big)\\
&= -(-1)^{|a|+|a||c|}\epsilon_5\big(\overline{(ab-(-1)^{|a||b|})c}\big)\\
&= 0.
\end{aligned}$$

□

We have a central extension

$$0 \longrightarrow \mathcal{W} \longrightarrow \mathfrak{st}(2,2,A)^\sharp \xrightarrow{\ \pi\ } \mathfrak{st}(2,2,A) \longrightarrow 0,$$

where $\mathfrak{st}(2,2,A)^\sharp = \mathfrak{st}(2,2,A) \oplus \mathcal{W}$ is the Lie superalgebra constructed by the surjective super 2-cocycle ψ, defined by the following relations:

$$a \mapsto F^{\sharp}_{ij}(a) \ \text{ is a } K\text{-linear map,} \tag{3.6.1}$$

$$[\mathcal{W},\mathcal{W}] = [F^{\sharp}_{ij}(a),\mathcal{W}] = 0, \tag{3.6.2}$$

$$[F^{\sharp}_{ij}(a), F^{\sharp}_{jk}(b)] = F^{\sharp}_{ik}(ab) \ \text{ for distinct i, j, k,} \tag{3.6.3}$$

$$[F^{\sharp}_{ij}(a), F^{\sharp}_{ij}(a)] = 0, \tag{3.6.4}$$

$$[F^{\sharp}_{ij}(a), F^{\sharp}_{ik}(b)] = 0, \tag{3.6.5}$$

$$[F^{\sharp}_{ij}(a), F^{\sharp}_{kl}(b)] = \epsilon_{\theta\big((i,j,k,l)\big)}(\overline{ab}) \ \text{ if } (i,j,k,l) \in P_1, P_2, P_3, P_4 \tag{3.6.6}$$

$$[F^{\sharp}_{ij}(a), F^{\sharp}_{kl}(b)] = (-1)^{|b|}\sigma\big((i,j,k,l)\big)\epsilon_{\theta\big((i,j,k,l)\big)}(\overline{ab}) \ \text{ if } (i,j,k,l) \in P_5, P_6. \tag{3.6.7}$$

Theorem 3.6.2. *The central extension* $0 \to \mathcal{W} \to \mathfrak{st}(2,2,A)^{\sharp} \to \mathfrak{st}(2,2,A)$ *is universal.*

Proof. Let

$$0 \longrightarrow \mathrm{V} \longrightarrow \widetilde{\mathfrak{st}}(2,2,A) \xrightarrow{\ \tau\ } \mathfrak{st}(2,2,A) \longrightarrow 0$$

be a central extension. As done in Theorem 3.4.1, we need a Lie superalgebra homomorphism $\rho\colon \mathfrak{st}(2,2,A)^{\sharp} \to \widetilde{\mathfrak{st}}(2,2,A)$ such that $\tau \circ \rho = \pi$. Choosing preimages $\widetilde{F}_{ij}(a)$ of τ, we have to check they satisfy relations (3.6.1)–(3.6.7). Doing the analogue computations as in Theorem 3.4.1 it is obvious that relations (3.6.1)–(3.6.5) are satisfied. We have to check that the $\widetilde{F}_{ij}(a)$ follow (3.6.6) and (3.6.7) to complete the proof.

As in the previous section, when i,j,k,l are distinct, denote

$$[\widetilde{F}_{ij}(a), \widetilde{F}_{kl}(1)] = v_{ijkl}(a).$$

To satisfy relations (3.6.6) and (3.6.7) we want to define the homomorphism from $\mathcal{W}$ by the expression $\rho(\epsilon_{\theta\big((i,j,k,l)\big)}(\overline{ab})) = \sigma\big((i,j,k,l)\big)v_{ijkl}(ab)$. If $(i,j,k,l) \in P_1,\dots,P_4$, we have to check the conditions

($\mathcal{R}$1) $2v_{ijkl}(a) = 0$,

($\mathcal{R}$2) $v_{ijkl}(a) = v_{kjil}(a) = v_{ilkj}(a) = v_{klij}(a)$,

($\mathcal{R}$3) $v_{ijkl}(a[b,c]) = 0$,

($\mathcal{R}$4) $[\widetilde{F}_{ij}(a), \widetilde{F}_{kl}(b)] = v_{ijkl}(ab)$.

Note that every permutation in $P_1, \dots, P_4$, has an element such that $|i| + |j| = \bar{0}$. Thus, recovering some computations of the previous section we have that

$$\begin{aligned} 0 &= \big[\widetilde{H}_{ij}(1,1), [\widetilde{F}_{ij}(a), \widetilde{F}_{kl}(1)]\big] \\ &= [\widetilde{F}_{ij}(a + (-1)^{|i|+|j|}a), \widetilde{F}_{kl}(1)] \\ &= v_{ijkl}(a + (-1)^{|i|+|j|}a). \end{aligned}$$

If $|i| + |j| = \bar{0}$, we have that $[\widetilde{F}_{ij}(a), \widetilde{F}_{kl}(1)] = -[\widetilde{F}_{ij}(a), \widetilde{F}_{kl}(1)]$. Then,

$$[\widetilde{F}_{il}(a), \widetilde{F}_{kj}(1)] = -(-1)^{(|k|+|l|)(|k|+|j|+|b|)}[\widetilde{F}_{ij}(a), \widetilde{F}_{kl}(1)],$$

so $[\widetilde{F}_{il}(a), \widetilde{F}_{kj}(1)] = -[\widetilde{F}_{il}(a), \widetilde{F}_{kj}(1)]$. Changing the indexes we obtain $(\mathcal{R}1)$ and $(\mathcal{R}2)$, and proceeding as in the proof of Theorem 3.5.2, conditions $(\mathcal{R}3)$ and $(\mathcal{R}4)$ are satisfied.

If $(i,j,k,l) \in P_5, P_6$, we have that

$$\begin{aligned} [\widetilde{F}_{ij}(a), \widetilde{F}_{kl}(b)] &= [\rho\big(F^{\sharp}_{ij}(a)\big), \rho\big(F^{\sharp}_{kl}(b)\big)] \\ &= \rho\big([F^{\sharp}_{ij}(a), F^{\sharp}_{kl}(b)]\big) \\ &= \rho\big((-1)^{|b|}\sigma\big((i,j,k,l)\big)\epsilon_{\theta\left((i,j,k,l)\right)}(\overline{ab})\big) \\ &= (-1)^{|b|}v_{ijkl}(ab) \\ &= (-1)^{|b|}[\widetilde{F}_{ij}(ab), \widetilde{F}_{kl}(1)]. \end{aligned}$$

Thus, we have to check the following conditions:

(C1) $v_{ijkl}(a) = -v_{kjil}(a) = -v_{ilkj}(a) = v_{klij}(a)$,

(C2) $[\widetilde{F}_{ij}(a), \widetilde{F}_{kl}(b)] = (-1)^{|b|}[\widetilde{F}_{ij}(ab), \widetilde{F}_{kl}(1)]$,

(C3) $v_{ijkl}(a[b,c]) = 0$.

To see (C1),

$$\begin{aligned} v_{ijkl}(a) &= [\widetilde{F}_{ij}(a), \widetilde{F}_{kl}(1)] \\ &= \big[\widetilde{F}_{ij}(a), [\widetilde{F}_{ki}(1), \widetilde{F}_{il}(1)]\big] = \big[[\widetilde{F}_{ij}(a), \widetilde{F}_{ki}(1)], \widetilde{F}_{il}(1)\big] \\ &= -(-1)^{(|i|+|j|+|a|)(|k|+|i|)}[\widetilde{F}_{kj}(a), \widetilde{F}_{il}(1)] \\ &= -[\widetilde{F}_{kj}(a), \widetilde{F}_{il}(1)] = -v_{kjil}(a), \end{aligned}$$

and

$$\begin{aligned} v_{ijkl}(a) &= [\widetilde{F}_{ij}(a), \widetilde{F}_{kl}(1)] \\ &= \big[\widetilde{F}_{ij}(a), [\widetilde{F}_{kj}(1), \widetilde{F}_{jl}(1)]\big] = \\ &= (-1)^{(|i|+|j|+|a|)(|k|+|j|)}\big[\widetilde{F}_{kj}(1), [\widetilde{F}_{ij}(a), \widetilde{F}_{jl}(1)]\big] \\ &= (-1)^{(|i|+|j|+|a|)(|k|+|j|)}[\widetilde{F}_{kj}(1), \widetilde{F}_{il}(a)] = \\ &= -(-1)^{(|k|+|j|)(|l|+|j|)}[\widetilde{F}_{il}(a), \widetilde{F}_{kj}(1)] = \\ &= -[\widetilde{F}_{il}(a), \widetilde{F}_{kj}(1)] = -v_{ilkj}(a). \end{aligned}$$

To check (C2),

$$\begin{aligned} [\widetilde{F}_{ij}(a), \widetilde{F}_{kl}(b)] &= \big[\widetilde{F}_{ij}(a), [\widetilde{F}_{kj}(1), \widetilde{F}_{jl}(b)]\big] \\ &= (-1)^{(|i|+|j|+|a|)(|k|+|j|)}\big[\widetilde{F}_{kj}(1), [\widetilde{F}_{ij}(a), \widetilde{F}_{jl}(b)]\big] \\ &= (-1)^{(|i|+|j|+|a|)(|k|+|j|)}\big[\widetilde{F}_{kj}(1), \widetilde{F}_{il}(ab)\big] \\ &\ - (-1)^{(|j|+|l|+|b|)(|k|+|j|)}[\widetilde{F}_{il}(ab), \widetilde{F}_{kj}(1)] \\ &= -(-1)^{|b|}[\widetilde{F}_{il}(ab), \widetilde{F}_{kj}(1)] = (-1)^{|b|}[\widetilde{F}_{ij}(ab), \widetilde{F}_{kl}(1)] \\ &= (-1)^{|b|}v_{ijkl}(ab), \end{aligned}$$

by part (C1).

Using (C2) and the fact that $|k| + |l| = \bar{1}$,

$$\begin{aligned} v_{ijkl}(a[b,c]) &= [\widetilde{F}_{ij}(a[b,c]), \widetilde{F}_{kl}(1)] \\ &= (-1)^{|b|+|c|}\sigma\big((i,j,k,l)\big)[\widetilde{F}_{ij}(a), \widetilde{F}_{kl}(bc - (-1)^{|b||c|}cb)] \\ &= (-1)^{|b|+|c|}\sigma\big((i,j,k,l)\big)\big[\widetilde{F}_{ij}(a), [\widetilde{H}_{kl}(b,c), \widetilde{F}_{kl}(1)]\big] \\ &= 0, \end{aligned}$$

by Jacobi identity, we have that (C3) is satisfied.

Thus, we obtained a Lie superalgebra homomorphism $\rho\colon \mathfrak{st}(2,2,A)^\sharp \to \widetilde{\mathfrak{st}}(2,2,A)$ completing the proof. □

Corollary 3.6.3. *The universal central extension of* $\mathfrak{sl}(2,2,A)$ *is* $\mathfrak{st}(2,2,A)^\sharp \cong \mathfrak{sl}(2,2,A) \oplus A_{\bar{2}}^4 \oplus A_{\bar{0}}^2$*. Moreover,* $\mathrm{H}_2\left(\mathfrak{st}(2,2,A)\right) \cong \mathcal{W} \cong A_{\bar{2}}^4 \oplus A_{\bar{0}}^2$.

3.7 Non-abelian tensor product and cyclic homology

In this section, we will consider the associative superalgebra A free as a K-supermodule. This assumption is needed in the definition of cyclic homology via complex.

Definition 3.7.1 ([10])**.** Let $\mathrm{C}_n(A) = A^{\otimes n}/I_n$ be the chain complex for $n \geqslant 0$ where I_n is the submodule generated by the relations

$$a_0 \otimes a_1 \otimes \cdots \otimes a_n - (-1)^{n+|a_0|\sum_{i=0}^{n-1}|a_i|} a_n \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1},$$

for $a_i \in A$ homogeneous. The boundary maps d_n are defined on generators by

$$\begin{aligned} d_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = &\sum_{i=0}^{n-1} (-1)^i a_0 \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes a_{i+1} \otimes \cdots a_n \\ &+ (-1)^{n+|a_n|\sum_{i=0}^{n-1}|a_i|} a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}, \end{aligned}$$

for $a_i \in A$ homogeneous. The *cyclic homology* $\mathrm{HC}_n(A)$ of the associative superalgebra A is the homology of the chain complex $\mathrm{C}_*(A)$.

In [7], it is introduced the non-abelian tensor product of two Lie superalgebras acting on each other. For the sake of simplicity, we recover the definition in the particular case of a Lie superalgebra L acting on itself by the canonical action, also called non-abelian tensor square.

Definition 3.7.2. Let L be a Lie superalgebra. The *non-abelian tensor square* $L \hat{\otimes} L$ is the tensor product of supermodules $L \otimes L$ quotient by the submodule generated by the relations

(i) $[x,y] \otimes z = x \otimes [y,z] - (-1)^{|x||y|} y \otimes [x,z]$,

(ii) $x \otimes [y,z] = (-1)^{|z|(|x|+|y|)}([z,x] \otimes y) - (-1)^{|x||y|}([y,x] \otimes z)$,

for all $x,y,z \in L$. It has a Lie superalgebra structure with bracket

$$[x \otimes y, z \otimes w] = [x,y] \otimes [z,w].$$

It is shown in [7] that if L is perfect, the homomorphism $\mathfrak{u}\colon L\widehat{\otimes}L \to L$, $x \otimes y \mapsto [x, y]$, is the universal central extension of L and $\mathsf{Ker}\,\mathfrak{u} = \mathrm{H}_2(L)$. Therefore, the universal central extension of $\mathfrak{sl}(m, n, A)$ is the same as the universal central extension of $\mathfrak{st}(m, n, A)$ which is $\mathfrak{st}(m, n, A)\widehat{\otimes}\mathfrak{st}(m, n, A)$. Additionally, we know that the universal central extension of $\mathfrak{st}(m, n, A)$ is just itself plus a K-supermodule, which will be denoted by $\mathcal{W}(m, n, A)$, possibly zero.

Theorem 3.7.3. *Let $m + n \geqslant 3$. Then there is an isomorphism of K-supermodules*

$$\mathrm{H}_2(\mathfrak{sl}(m, n, A)) \cong \mathrm{HC}_1(A) \oplus \mathcal{W}(m, n, A).$$

Proof. We consider the following diagram

$$\begin{array}{ccccccccc} 0 \to & \mathrm{HC}_1(A) \oplus \mathcal{W}(m, n, D) & \longrightarrow & \dfrac{A \otimes A}{\mathrm{Im}\, d_2} \oplus \mathcal{W}(m, n, A) & \xrightarrow{d_1} & [A, A] & \longrightarrow 0 \\ & & & \mathrm{Str}_2 \uparrow\downarrow \mu & & \mathrm{Str}_1 \uparrow\downarrow E_{11}(-) & \\ 0 \longrightarrow & \mathrm{H}_2\big(\mathfrak{st}(m, n, A)\big) & \longrightarrow & \mathfrak{st}(m, n, A)\widehat{\otimes}\mathfrak{st}(m, n, A) & \xrightarrow{\omega} & \mathfrak{sl}(m, n, A) & \to 0, \end{array}$$

where $\mu(a \otimes b) = F_{1j}(a) \otimes F_{j1}(b) - (-1)^{|a||b|} F_{1j}(ba) \otimes F_{j1}(1)$, $\mu\big(v_{ijkl}(a)\big) = F_{ij}(a) \otimes F_{kl}(b)$ and

$$\mathrm{Str}_2\left(F_{ij}(a) \otimes F_{kl}(b)\right) = \begin{cases} a \otimes b, & \text{if } i = j \text{ and } k = l, \\ v_{ijkl}(ab), & \text{when it makes sense depending of } m, n, \\ 0, & \text{otherwise.} \end{cases}$$

It is a straightforward computation that $\mu \circ \mathrm{Str}_2$ and $E_{11}(-) \circ \mathrm{Str}_1$ are the identity maps and that the diagram is commutative. Then the restrictions of Str_2 to the kernel of ω is also a split epimorphism, with μ restricted to the kernel of d_1 as section. Let us see that these restrictions are indeed isomorphisms. An element in the kernel of ω, is a sum of elements of the form $F_{ij}(a) \otimes F_{ji}(b)$ plus the elements of $\mathcal{W}(m, n, D)$. Any element of $\mathsf{Ker}\,\omega$ can be written as an element of $\mathsf{Im}\,\mu$ plus $\sum_{i=2}^{m+n} F_{1i}(a_i) \otimes F_{i1}(1)$, since

$$F_{ij}(a) \otimes F_{ji}(b) = F_{i1}(a) \otimes F_{1i}(b) - (-1)^{(|F_{ij}(a)|)(|F_{ji}(b)|)} F_{j1}(ba) \otimes F_{1j}(1),$$

and

$$\begin{aligned} F_{1j}(a) \otimes F_{j1}(b) = & F_{1j}(a) \otimes F_{j1}(b) - (-1)^{|a||b|} F_{j1}(ba) \otimes F_{1j}(1) \\ & + (-1)^{|a||b|} F_{j1}(ba) \otimes F_{1j}(1). \end{aligned}$$

Furthermore, if it is in the kernel of ω, all the a_i must be zero. Then the restriction of μ to the kernel of d_1 is surjective. □

3.8 Concluding remarks

Combining the main theorems presented here with the main theorems of [3, 7] we have a complete characterization of $\mathrm{H}_2\left(\mathfrak{st}(m,n,A)\right)$ and $\mathrm{H}_2\left(\mathfrak{sl}(m,n,A)\right)$ for $m+n \geqslant 3$.

Theorem 3.8.1. *Let K a unital commutative ring and A an associative unital K-superalgebra. Then,*

$$\mathrm{H}_2\left(\mathfrak{st}(m,n,A)\right) = \begin{cases} 0 & \textit{for } m+n \geqslant 5 \textit{ or } m=2, n=1,\\ A_3^6 & \textit{for } m=3, n=0,\\ A_2^6 & \textit{for } m=4, n=0,\\ \Pi(A_2)^6 & \textit{for } m=3, n=1,\\ A_2^4 \oplus A_0^2 & \textit{for } m=2, n=2,\end{cases}$$

where A_m is the quotient of A by the ideal $mA + A[A,A]$ (Definition 3.2.3) and Π is the parity change functor.

Theorem 3.8.2. *Let K a unital commutative ring and A an associative unital K-superalgebra with a K-basis containing the identity. Then,*

$$\mathrm{H}_2\left(\mathfrak{sl}(m,n,A)\right) = \begin{cases} \mathrm{HC}_1(A) & \textit{for } m+n \geqslant 5 \textit{ or } m=2, n=1,\\ \mathrm{HC}_1(A) \oplus A_3^6 & \textit{for } m=3, n=0,\\ \mathrm{HC}_1(A) \oplus A_2^6 & \textit{for } m=4, n=0,\\ \mathrm{HC}_1(A) \oplus \Pi(A_2)^6 & \textit{for } m=3, n=1,\\ \mathrm{HC}_1(A) \oplus A_2^4 \oplus A_0^2 & \textit{for } m=2, n=2,\end{cases}$$

where A_m is the quotient of A by the ideal $mA + A[A,A]$ (Definition 3.2.3) and Π is the parity change functor.

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Chapter 4

Universal central extensions of Leibniz superalgebras over superdialgebras

Abstract

We complete the problem of finding the universal central extension in the category of Leibniz superalgebras of $\mathfrak{sl}(m, n, D)$ when $m + n \geqslant 3$ and D is a superdialgebra, solving in particular the problem when D is an associative algebra, superalgebra or dialgebra. To accomplish this task we use a different method than the standard studied in the literature. We introduce and use the non-abelian tensor square of Leibniz superalgebras and its relations with the universal central extension.

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4.1 Introduction

Leibniz algebras, the non-antisymmetric analogue of Lie algebras, were first defined by Bloh [1] and later recovered by Loday in [22] when he handled periodicity phenomena in algebraic K-theory. Many authors have studied this structure and it has some interesting applications in Geometry and Physics ([16], [25], [6]). On the other hand, the theory of superalgebras arises directly from supersymmetry, a part of the theory of elemental particles, in order to have a better understanding of the geometrical structure of spacetime and to complete the substantial meaningful task of the unification of quantum theory and general relativity ([32]). The study of Lie or Leibniz superalgebras has been a very active field in the recent years since the classification of simple complex finite-dimensional Lie superalgebras by Kac in [14].

The study of central extensions is a very important topic in mathematics. There is a direct connection between central extensions and (co)homology, and they also have relations with Physics ([30]). In particular, universal central extensions have been studied in many different structures as groups [27], Lie algebras [11], [31] or Lie superalgebras [28]. A very interesting tool in the study of universal central extensions is the non-abelian tensor product introduced in [2] and extended to Lie algebras in [5] and to Lie superalgebras in [9].

The theory related with the universal central extension of the special linear algebra $\mathfrak{sl}(n, A)$ has been very active due its relation with cyclic homology and its relevance in algebraic K-theory. The first approach was in the category of Lie algebras by Kassel and Loday in [15] where they described it when $n \geqslant 5$ and A is an associative algebra, and in [8] it was obtained for $n \geqslant 3$. For the Lie superalgebra $\mathfrak{sl}(n, A)$ and A an associative superalgebra, the universal central extension was given in [3]. For the special linear superalgebra $\mathfrak{sl}(m, n, A)$, the universal central extension for A an associative algebra was found in [26] and [29]; and for A an associative superalgebra was found in [4] and [10]. In the category of Leibniz algebras, the universal central extension of $\mathfrak{sl}(n, A)$ (seen as a Leibniz algebra), where A is an associative algebra, was found in [24] when $n \geqslant 5$ and in [13] when $n \geqslant 3$. For the Leibniz superalgebra $\mathfrak{sl}(m, n, A)$, when $m + n \geqslant 5$ and A is an associative algebra it was calculated in [21]. In [18] it was found for the Leibniz algebra $\mathfrak{sl}(m, D)$ and for the Leibniz superalgebra $\mathfrak{sl}(m, n, D)$, where $m \geqslant 5$ and $m + n \geqslant 5$, respectively, and D is an associative dialgebra.

The aim of this paper is to complete the task, finding the universal central extension of $\mathfrak{sl}(m, n, D)$ where D is a superdialgebra and $m + n \geqslant 3$. Since associative algebras, associative superalgebras and dialgebras are all examples of associative superdialgebras, we will solve all cases at once. Moreover, we obtain a result contradicting a specific point of a theorem given in [18]. The most interesting part of this paper is that the method used is not the same as in all the papers cited above. Due its relation with central extensions, we introduce and use the non-abelian tensor square of Leibniz superalgebras providing another point of view to this topic.

4.2 Preliminaries

Throughout the paper we fix a commutative ring R with unit.

4.2.1 Dialgebras

We recall from [23] the definitions and basic examples of (super)dialgebras.

Definition 4.2.1. An *associative dialgebra* (*dialgebra* for short) is an R-module equipped with two R-linear maps

$$\vdash : D \otimes_R D \to D,$$
$$\dashv : D \otimes_R D \to D,$$

where $\vdash$ and $\dashv$ are associative and satisfy the following conditions:

$$\begin{cases} a \dashv (b \dashv c) = a \dashv (b \vdash c), \\ (a \vdash b) \dashv c = a \vdash (b \dashv c), \\ (a \dashv b) \vdash c = (a \vdash b) \vdash c, \end{cases}$$

for all $a, b, c \in D$.

An *ideal* $I \subset D$ is an R-submodule such that if a or b belong to I then $a \dashv b \in D$ and $a \vdash b \in D$.

A *bar-unit* in D is an element $e \in D$ such that for all $a \in D$,

$$a \dashv e = a = e \vdash a.$$

Note that a bar-unit may not be unique. A *unital dialgebra* is a dialgebra with a chosen bar-unit, that will be denoted by 1.

An *associative superdialgebra* (*superdialgebra* for short) is a dialgebra equipped with a $\mathbb{Z}$-graded structure compatible with the two operations, i.e. $D_{\bar{\alpha}} \vdash D_{\bar{\beta}} \subseteq D_{\bar{\alpha}+\bar{\beta}}$ and $D_{\bar{\alpha}} \dashv D_{\bar{\beta}} \subseteq D_{\bar{\alpha}+\bar{\beta}}$, for $\bar{\alpha}, \bar{\beta} \in \mathbb{Z}$. The concepts of bar-unit, unital and ideal are analogous in superdialgebras. Note that the bar-unit is always even.

Example 4.2.2. We introduce some examples of superdialgebras.

(i) An associative (super)algebra defines a (super)dialgebra structure in a canonical way, where $a \dashv b = ab = a \vdash b$. If it is unital, then the superdialgebra is unital.

(ii) Let (A, d) a differential associative (super)algebra, i.e., $d(ab) = d(a)b + ad(b)$ and $d^2 = 0$. We define the two operations by

$$a \dashv b = ad(b),$$
$$a \vdash b = d(a)b.$$

It is immediate to check that with these operations (A, d) is a (super)dialgebra.

(iii) Let A an associative (super)algebra, M an A-(super)bimodule and $f\colon M \to A$ an A-(super)bimodule map. Then we can define a (super)dialgebra structure with operations

$$m \dashv m' = mf(m'),$$
$$m \vdash m' = f(m)m'.$$

(iv) Let D and D' be two superdialgebras. The tensor product $D \otimes_R D'$ is a superdialgebra where

$$(a \otimes a') \dashv (b \otimes b') = (-1)^{|a'||b|}(a \dashv b) \otimes (a' \dashv b'),$$
$$(a \otimes a') \vdash (b \otimes b') = (-1)^{|a'||b|}(a \vdash b) \otimes (a' \vdash b').$$

4.2.2 Leibniz superalgebras

Definition 4.2.3. A *Leibniz superalgebra* L is an R-supermodule with an R-linear even map

$$[-,-]\colon L \otimes_R L \to L,$$

satisfying the *Leibniz identity*

$$\big[x, [y, z]\big] = \big[[x, y], z\big] - (-1)^{|y||z|}\big[[x, z], y\big],$$

for all $x, y, z \in L$.

Note that a Leibniz superalgebra where the identity $[x,y] = -(-1)^{|x||y|}[y,x]$ also holds, is a Lie superalgebra.

Example 4.2.4.

(i) A Lie superalgebra is in particular a Leibniz superalgebra.

(ii) Let D be a superdialgebra. Then D with the bracket

$$[a,b] = a \dashv b - (-1)^{|a||b|} b \vdash a,$$

is a Leibniz superalgebra. If the two operations $\dashv$ and $\vdash$ are equal, i.e., D is also an associative superalgebra, this bracket also induces a Lie superalgebra structure.

Definition 4.2.5. The *centre* of a Leibniz superalgebra L, denoted by $\mathrm{Z}(L)$, is the ideal formed by the elements $z \in L$ such that $[z,x] = [x,z] = 0$ for all $x \in L$. The *commutator* of L, denoted by $[L,L]$, is the ideal generated by the elements $[x,y]$ where $x,y \in L$. A Leibniz superalgebra is called *perfect* if $L = [L,L]$.

Definition 4.2.6. A *central extension* of a Leibniz superalgebra L is a surjective homomorphism $\phi\colon M \to L$ such that $\mathsf{Ker}\,\phi \subseteq \mathrm{Z}(M)$. We say that a central extension $\mathfrak{u}\colon U \to L$ is *universal* if for any central extension $\phi\colon M \to L$ there is a unique homomorphism $f\colon U \to M$ such that $\mathfrak{u} = \phi \circ f$.

The theory of central extensions of Leibniz superalgebras is studied in [20]. We obtain the following straightforward results.

Proposition 4.2.7. *Let $\phi\colon E \to M$ and $\psi\colon M \to L$ be two central extensions of Leibniz superalgebras. Then ϕ is universal if and only if $\psi \circ \phi$ is universal.*

Proposition 4.2.8. *Let M be a Leibniz superalgebra and L an R-supermodule. An R-supermodule homomorphism $\varphi\colon M \to L$ such that $\mathsf{Ker}\,\varphi \subseteq \mathsf{Z}(M)$ defines a Leibniz superalgebra structure in L where the bracket is given by*

$$[x,y] = \varphi([\varphi^{-1}(x), \varphi^{-1}(y)]).$$

for $x,y \in L$.

Now, we introduce the homology of Leibniz superalgebras with trivial coefficients adapting it from the non-graded version [24].

Definition 4.2.9. Let L be a Leibniz superalgebra and $\delta_n\colon L^{\otimes n} \to L^{\otimes n-1}$ the R-linear map defined on generators by

$$\delta_n(x_1 \otimes \cdots \otimes x_n) = \sum_{i<j} (-1)^{n-j+|x_j|(|x_{i+1}|+\cdots+|x_{j-1}|)} x_1 \otimes \cdots \otimes x_{i-1} \otimes [x_i, x_j] \otimes x_{i+1} \otimes \cdots \otimes \widehat{x_j} \otimes \cdots \otimes x_n.$$

The *homology of Leibniz superalgebras* with trivial coefficients is the homology of the chain complex formed by δ_n, i.e.

$$\mathrm{HL}_n(L) = \frac{\mathsf{Ker}\, \delta_n}{\mathsf{Im}\, \delta_{n+1}}$$

Note that $\delta_3(x \otimes y \otimes z) = -[x, y] \otimes z + x \otimes [y, z] + (-1)^{|y||z|}[x, z] \otimes y$.

In [12] it is defined a non-abelian tensor product of Leibniz algebras and in [17] a variation is introduced. Both coincide in the case of perfect Leibniz algebras, so for simplicity we will generalize to Leibniz superalgebras the version of [17].

Definition 4.2.10. Let L be a perfect Leibniz superalgebra. The *non-abelian tensor product* of L is

$$L \otimes L = \frac{L \otimes_R L}{\mathsf{Im}\, \delta_3},$$

where δ_3 is the map defined on the chain complex of Leibniz homology and the bracket is $[x \otimes y, x' \otimes y'] = [x, y] \otimes [x', y']$. Therefore, we have a short exact sequence.

$$0 \longrightarrow \mathrm{HL}_2(L) \longrightarrow L \otimes L \xrightarrow{\delta_2} L \longrightarrow 0.$$

Theorem 4.2.11. *Let L be a perfect Leibniz superalgebra. Then $\delta_2\colon L \otimes L \to L$ is the universal central extension of L and its kernel is $\mathrm{HL}_2(L)$.*

Proof. Let $\sum_i x_i \otimes y_i$ be in the kernel of δ_2. Then $\sum_i [x_i, y_i] = 0$, so $[\sum_i x_i \otimes y_i, x' \otimes y'] = \sum_i [x_i, y_i] \otimes [x', y'] = 0$. Therefore, δ_2 is a central extension. Let $0 \longrightarrow K \xrightarrow{\iota} M \xrightarrow{\phi} L \longrightarrow 0$ be a central extension. We define a homomorphism $\mathfrak{u}\colon L \otimes L \to M$, $x \otimes y \mapsto [\bar{x}, \bar{y}]$, where $\bar{x}$ and $\bar{y}$ are preimages by ϕ of x and y respectively. This homomorphism is well defined since $\mathsf{Ker}\, \phi \subseteq \mathrm{Z}(M)$. If $\mathfrak{u}, \mathfrak{u}'$ are two homomorphisms such that $\phi \circ \mathfrak{u} = \phi \circ \mathfrak{u}'$, then $\mathfrak{u} - \mathfrak{u}' = \iota \circ \eta$ where $\eta\colon L \otimes L \to K$ and $\eta([L \otimes L, L \otimes L]) = 0$. Since L is perfect, $L \otimes L$ is also perfect and $\mathfrak{u}$ is unique. □

4.2.3 Matrix Leibniz superalgebras

Let $\{1, \dots, m\} \cup \{m+1, \dots, m+n\}$ be a graded set and $D = D_{\bar{0}} \oplus D_{\bar{1}}$ unital superdialgebra. We consider the set $\mathcal{M}(m, n, D)$ of $(m+n) \times (m+n)$-matrices. Let $E_{ij}(a)$ be the matrix with $a \in D$ in the position (i, j) and zeros elsewhere. We define a grading in $\mathcal{M}(m, n, D)$ where the homogeneous elements are $E_{ij}(a)$ with a homogeneous and the grading is given by $|E_{ij}(a)| = |i| + |j| + |a|$. Now we define the *general Leibniz superalgebra* $\mathfrak{gl}(m, n, D)$ which has $\mathcal{M}(m, n, D)$ with the previous grading as underlying set and the Leibniz bracket is given by $[x, y] = x \dashv y - (-1)^{|x||y|} y \vdash x$. If $m + n \geqslant 2$, we define the *special linear Leibniz superalgebra* $\mathfrak{sl}(m, n, D) = [\mathfrak{gl}(m, n, D), \mathfrak{gl}(m, n, D)]$. It is easy to see that $\mathfrak{sl}(m, n, D)$ is generated by $E_{ij}(a)$ with $a \in D_{\bar{0}} \cup D_{\bar{1}}$ and $1 \leqslant i \neq j \leqslant m+n$ and the bracket is given by

$$[E_{ij}(a), E_{kl}(b)] = \delta_{jk} E_{il}(a \dashv b) - (-1)^{|E_{ij}(a)||E_{kl}(b)|} \delta_{il} E_{kj}(b \vdash a).$$

Following [18], if $m+n \geqslant 3$ then $\mathfrak{sl}(m, n, D)$ is perfect. We define the *supertrace* as the R-linear homomorphism $\mathrm{Str}_1 \colon \mathfrak{gl}(m, n, D) \to D$ with

$$\mathrm{Str}_1(x) = \sum_{i=1}^{m+n} (-1)^{|i|(|i|+|x_{ii}|)} x_{ii}.$$

Note that $\mathfrak{sl}(m, n, D) = \{x \in \mathfrak{gl}(m, n, D) : \mathrm{Str}_1(x) \in [D, D]\}$.

Definition 4.2.12. Let D be a superdialgebra and m and n non-negative integers such that $m + n \geqslant 3$. We define the *Steinberg Leibniz superalgebra* denoted by $\mathfrak{stl}(m, n, D)$ as the Leibniz superalgebra generated by the elements $F_{ij}(a)$ with $a \in D_{\bar{0}} \cup D_{\bar{1}}$, $1 \leqslant i \neq j \leqslant m + n$, where the grading is given by $|F_{ij}(a)| = |i| + |j| + |a|$, subject to the relations

$$\begin{aligned}
&a \mapsto F_{ij}(a) \text{ is } R\text{-linear}, && \\
&[F_{ij}(a), F_{kl}(b)] = F_{il}(a \dashv b), && \text{if } i \neq l \text{ and } j = k, \\
&[F_{ij}(a), F_{kl}(b)] = -(-1)^{|F_{ij}(a)||F_{kl}(b)|} F_{kj}(b \vdash a), && \text{if } i = l \text{ and } j \neq k, \\
&[F_{ij}(a), F_{kl}(b)] = 0, && \text{if } i \neq l \text{ and } j \neq k.
\end{aligned}$$

We recall from [19] that $\mathfrak{stl}(m, n, D)$ is perfect and the canonical Leibniz superalgebra homomorphism $\phi \colon \mathfrak{stl}(m, n, D) \to \mathfrak{sl}(m, n, D)$, $F_{ij}(a) \mapsto E_{ij}(a)$ is a central extension.

4.3 Universal central extension of $\mathfrak{sl}(m, n, D)$

In this section we are going to show that $\mathfrak{stl}(m, n, D)$ is the universal central extension of $\mathfrak{sl}(m, n, D)$ when $m + n \geqslant 5$. We are going to use a slightly different method than usual found in the literature. The strategy is to prove that there is an isomorphism between the non-abelian tensor product $\mathfrak{stl}(m, n, D) \otimes \mathfrak{stl}(m, n, D)$ and $\mathfrak{stl}(m, n, D)$ itself. Then Proposition 4.2.7 implies that $\mathfrak{stl}(m, n, D)$ is the universal central extension of $\mathfrak{sl}(m, n, D)$.

Theorem 4.3.1. *There is an isomorphism* $\mathfrak{stl}(m, n, D) \otimes \mathfrak{stl}(m, n, D) \cong \mathfrak{stl}(m, n, D)$ *for* $m + n \geqslant 5$.

Proof. Let us consider the following homomorphisms defined on generators:

$$\varphi \colon \mathfrak{stl}(m, n, D) \otimes \mathfrak{stl}(m, n, D) \to \mathfrak{stl}(m, n, D), F_{ij}(a) \otimes F_{kl}(b) \mapsto [F_{ij}(a), F_{kl}(b)],$$
$$\psi \colon \mathfrak{stl}(m, n, D) \to \mathfrak{stl}(m, n, D) \otimes \mathfrak{stl}(m, n, D), F_{ij}(a) \mapsto F_{ik}(a) \otimes F_{kj}(1).$$

It is straightforward that φ is a well-defined Leibniz superalgebra homomorphism. For different i, j, k we have

$$F_{ik}(a) \otimes F_{kj}(1) = [F_{is}(a), F_{sk}(1)] \otimes F_{kj}(1) = F_{is}(a) \otimes F_{sj}(1),$$

so ψ does not depend of the choice of k. To check if ψ preserves the relations it is enough to see that:

(a) If $i \neq l$ and $j = k$,

$$F_{ij}(a) \otimes F_{kl}(b) = F_{ij}(a) \otimes [F_{ks}(b), F_{sl}(1)] = F_{is}(a \dashv b) \otimes F_{sl}(1).$$

(b) If $i = l$ and $j \neq k$,

$$\begin{aligned} F_{ij}(a) \otimes F_{kl}(b) &= [F_{is}(a), F_{sj}(1)] \otimes F_{kl}(b) \\ &= -(-1)^{(|i|+|j|+|a|)(|k|+|ss|+|b|)} F_{ks}(b \vdash a) \otimes F_{sj}(1). \end{aligned}$$

(c) If $i \neq l$ and $j \neq k$,

$$F_{ij}(a) \otimes F_{kl}(b) = [F_{is}(a), F_{sj}(1)] \otimes F_{kl}(b) = 0.$$

Moreover, these relations show that $\psi \circ \phi$ is the identity map and it is obvious that $\phi \circ \psi$ is the identity map too. □

4.4 Universal central extension of $\mathfrak{sl}(m,n,D)$ when $m+n<5$

In this section we will find the universal central extension of $\mathfrak{sl}(m,n,D)$ when $3 \leqslant m+n < 5$. We need some preliminary results first. Recall that $[D,D]$ is the subalgebra generated by the elements $a \dashv b - (-1)^{|a||b|} b \vdash a$. It happens that in superdialgebras, this is not necessarily an ideal.

Lemma 4.4.1. *Let D be a unital superdialgebra. We have that $D \dashv [D,D] \subseteq [D,D] \dashv D$, $[D,D] \vdash D \subseteq D \vdash [D,D]$ and $[D,D] \dashv D = D \vdash [D,D]$. Then the ideal generated by the elements $a \dashv b - (-1)^{|a||b|} b \vdash a$ is just $[D,D] \dashv D$.*

Proof. The results follow, respectively, from the identities

$$a \dashv [b,c] = [a,b] \dashv c - (-1)^{|b||c|}[a \dashv c, b] \dashv 1,$$
$$[a,b] \vdash c = -(-1)^{|b||c|} a \vdash [c,b] + (-1)^{|b||c|} 1 \vdash [a \vdash c, b],$$
$$[a,b] \dashv c = -(-1)^{|a||b|} b \vdash [a,c] + [a, b \vdash c].$$

□

Definition 4.4.2. Let D be a superdialgebra and m a positive integer. Let $\mathcal{I}_m$ be the ideal of D generated by the elements ma and $a \dashv b-(-1)^{|a||b|} b \vdash a$. We denote the quotient

$$D_m = \frac{D}{\mathcal{I}_m}.$$

We claim that $\mathfrak{stl}(m,n,D) \otimes \mathfrak{stl}(m,n,D) \cong \mathfrak{stl}(m,n,D) \oplus \mathcal{W}(m,n,D)$ where $\mathcal{W}(m,n,D)$ is an R-supermodule which depends on m and n and the Leibniz superalgebra structure is given by an R-supermodule homomorphism $\varphi\colon \mathfrak{stl}(m,n,D) \otimes \mathfrak{stl}(m,n,D) \to \mathfrak{stl}(m,n,D) \oplus \mathcal{W}(m,n,D)$ in the conditions of Proposition 4.2.8. Then we will define an inverse.

4.4.1 Case of $\mathfrak{sl}(4,0,D)$

Let $\mathcal{W}(4,0,D)$ be the direct sum of six copies of D_2. The elements will be represented by $v_{ijkl}(a)$ where $1 \leqslant i,j,k,l \leqslant 4$ are distinct, $a \in D$ and $|v_{ijkl}(a)| = |a|$. They will be related by R-linearity, the equivalence relations of D_2 and by $v_{ijkl}(a) = -v_{ilkj}(a) = -v_{kjil}(a) = v_{klij}(a)$.

Theorem 4.4.3. *The universal central extension of* $\mathfrak{sl}(4,0,D)$ *is* $\mathfrak{stl}(4,0,D) \oplus D_2^6$.

Proof. Let $\varphi\colon \mathfrak{stl}(4,0,D) \otimes \mathfrak{stl}(4,0,D) \to \mathfrak{stl}(4,0,D) \oplus D_2^6$ be the homomorphism defined on generators by $F_{ij}(a) \otimes F_{kl}(b) \mapsto v_{ijkl}(a \dashv b)$, if i,j,k,l are distinct and $F_{ij}(a) \otimes F_{kl}(b) \mapsto [F_{ij}(a), F_{kl}(b)]$, otherwise. It is obvious that it conserves the grading and that the kernel is inside the centre, so we have to check if φ sends the relation of the non-abelian tensor product to zero.

The relation on generators is given by

$$F_{ij}(a) \otimes [F_{kl}(b), F_{st}(c)] - [F_{ij}(a), F_{kl}(b)] \otimes F_{st}(c) + (-1)^{|F_{kl}(b)||F_{st}(c)|}[F_{ij}(a), F_{st}(c)] \otimes F_{kl}(b). \tag{Gen}$$

If is not involved any preimage of $\mathcal{W}(4,0,D)$, then the image is just the Leibniz identity on $\mathfrak{stl}(4,0,D)$. To have any $v_{ijkl}(a)$ we need that in i,j,k,l,s,t one element appears three times and the others three once. Using the relation $v_{ijkl}(a \dashv [b,c]) = 0 = v_{ijkl}([a,b] \dashv c)$ and that we do not need to worry about signs ($v_{ijkl}(2a) = 0$) it is easy to go through the different possibilities and check that they all vanish. Therefore, the bracket defined on $\mathfrak{stl}(m,n,D) \oplus D_2^6$ is the standard bracket unless if i,j,k,l are distinct, then $[F_{ij}(a), F_{kl}(b)] = v_{ijkl}(a \dashv b)$. Moreover, the elements $v_{ijkl}(a)$ are in the centre.

Now we define $\psi\colon \mathfrak{stl}(4,0,D) \oplus D_2^6 \to \mathfrak{stl}(4,0,D) \otimes \mathfrak{stl}(4,0,D)$ by $F_{ij}(a) \mapsto F_{ik}(a) \otimes F_{kj}(1)$ and $v_{ijkl}(a) \mapsto F_{ij}(a) \otimes F_{kl}(1)$. It is well defined for the elements of $\mathfrak{stl}(4,0,D)$ (as in Theorem 4.3.1) so we have to check if it is well defined for the elements of D_2^6.

$$F_{ij}(a) \otimes F_{kl}(1) = [F_{il}(a), F_{lj}(1)] \otimes F_{kl}(1) = -F_{il}(a) \otimes F_{kj}(1),$$
$$F_{ij}(a) \otimes F_{kl}(1) = F_{ij}(a) \otimes [F_{ki}(1), F_{il}(1)] = -F_{kj}(a) \otimes F_{il}(1).$$

So $v_{ijkl}(a) = -v_{ilkj}(a) = -v_{kjil}(a) = v_{klij}(a)$. Now,

$$\begin{aligned} 0 &= [F_{ij}(a) \otimes F_{ji}(b), F_{ij}(c) \otimes F_{kl}(1)] = [F_{ij}(a), F_{ji}(b)] \otimes [F_{ij}(c), F_{kl}(1)] \\ &= \big[[F_{ij}(a), F_{ji}(b)], F_{ij}(c)\big] \otimes F_{kl}(1) \\ &= \big[[F_{ij}(a), F_{ji}(b)], [F_{ik}(c), F_{kj}(1)]\big] \otimes F_{kl}(1) \\ &= F_{ij}(a \dashv b \dashv c + (-1)^{|a||b|+|a||c|+|b||c|} c \vdash b \vdash a) \otimes F_{kl}(1), \end{aligned}$$

Choosing $b = c = 1$ we have $F_{ij}(2a) \otimes F_{kl}(1) = 0$. Choosing $c = 1$, we have

$$F_{ij}(a \dashv b + (-1)^{|a||b|} b \vdash a) \otimes F_{kl}(1) = 0.$$

Therefore, ψ is a well-defined R-supermodule homomorphism. Moreover, the identity

$$F_{ij}(a)\otimes F_{kl}(b) = F_{ij}(a)\otimes [F_{ki}(b), F_{il}(1)] = -(-1)^{|a||b|}F_{kj}(b\vdash a)\otimes F_{il}(1)$$
$$= F_{kj}(a\dashv b)\otimes F_{il}(1),$$

shows that ψ is a Leibniz superalgebra homomorphism and that φ and ψ are inverses to each other. □

4.4.2 Case of $\mathfrak{sl}(3,1,D)$

Let $\mathcal{W}(3,1,D)$ be the direct sum of six copies of $\Pi(D_2)$, where Π denotes the parity change functor. The elements will be represented by $v_{ijkl}(a)$ with the same relations as in the previous case.

Theorem 4.4.4. *The universal central extension of* $\mathfrak{sl}(3,1,D)$ *is* $\mathfrak{stl}(3,1,D)\oplus \Pi(D_2)^6$.

Proof. Assuming that $|i|=0$, we can adapt the proof of Theorem 4.4.3. In the case that $|i|=1$,

$$0 = [F_{kl}(a)\otimes F_{lk}(1), F_{ij}(1)\otimes F_{kl}(1)],$$

gives us that $F_{kl}(2a)\otimes F_{ij}(1) = 0$, and we adapt again the proof of Theorem 4.4.3. □

4.4.3 Case of $\mathfrak{sl}(2,2,D)$

Let $\mathcal{W}(2,2,D)$ be the direct sum of four copies of D_2 and two copies of D_0. The elements will be represented by $v_{ijkl}(a)$ related by $v_{ijkl}(a) = -v_{ilkj}(a) = -v_{kjil}(a) = v_{klij}(a)$, where $v_{1324}(a)$ and $v_{3142}(a)$ will represent the copies of D_0 and the rest will be the copies of D_2. Note that $v_{ijkl}(a)$ represents one copy of D_0 if and only if $|i|+|j| = \bar{1} = |k|+|ll|$ and $|i|+|k| = \bar{0} = |j|+|ll|$.

Theorem 4.4.5. *The universal central extension of* $\mathfrak{sl}(2,2,D)$ *is* $\mathfrak{stl}(2,2,D)\oplus D_2^4\oplus D_0^2$.

Proof. Let S_4 be the group of permutations of 4 elements and let $\sigma\colon S_4\to\{-1,1\}$ be the map that sends $(1423),(2314),(3241),(4132)$ to -1 and the rest to 1. Note that if $\sigma(ijkl) = -1$, then $v_{ijkl}(a)$ represents a copy of D_0. Let us consider the homomorphism

$$\varphi\colon \mathfrak{stl}(2,2,D) \otimes \mathfrak{stl}(2,2,D) \to \mathfrak{stl}(2,2,D) \oplus D_2^4 \oplus D_0^2,$$

defined on generators by

$$F_{ij}(a) \otimes F_{kl}(b) \mapsto \begin{cases} (-1)^b \sigma(ijkl) v_{ijkl}(a \dashv b) & \text{if } i,\ j,\ k,\ l \text{ are distinct,} \\ [F_{ij}(a), F_{kl}(b)] & \text{otherwise.} \end{cases}$$

Again we need to check that φ sends the relation of the non-abelian tensor product to zero. If $v_{ijkl}(a)$ represents an element of D_2, then the proof is similar as the proof given in Theorem 4.4.3. Therefore, we have to check if relation (Gen) vanishes when an element of D_0 appears. Avoiding symmetries, the choices that we have to check are $(i,j,k,l,s,t) = (1,3,2,1,1,4)$, $(1,3,2,3,3,4)$, $(1,3,1,4,2,1)$ and $(1,3,3,4,2,3)$. It is a straightforward computation and we omit it.

Now we define $\psi\colon \mathfrak{stl}(2,2,D) \oplus D_2^4 \oplus D_0^2 \to \mathfrak{stl}(2,2,D) \otimes \mathfrak{stl}(2,2,D)$ by $F_{ij}(a) \mapsto F_{ik}(a) \otimes F_{kj}(1)$ and $v_{ijkl}(a) \mapsto \sigma(ijkl) F_{ij}(a) \otimes F_{kl}(1)$. To check that is a well-defined homomorphism we can follow the proofs of Theorem 4.3.1 and Theorem 4.4.3 and we will cover all the cases except the two copies of D_0. In that case, we have that

$$F_{ij}(a) \otimes F_{kl}(1) = [F_{il}(a), F_{lj}(1)] \otimes F_{kl}(1) = -F_{il}(a) \otimes F_{kj}(1),$$
$$F_{ij}(a) \otimes F_{kl}(1) = F_{ij}(a) \otimes [F_{ki}(1), F_{il}(1)] = -F_{kj}(a) \otimes F_{il}(1).$$

So $v_{ijkl}(a) = -v_{ilkj}(a) = -v_{kjil}(a) = v_{klij}(a)$. Then,

$$\begin{aligned} 0 &= [F_{ij}(a) \otimes F_{ji}(b), F_{ij}(c) \otimes F_{kl}(1)] = [F_{ij}(a), F_{ji}(b)] \otimes [F_{ij}(c), F_{kl}(1)] \\ &= F_{ij}(a \dashv b \dashv c + (-1)^{(|a|+\bar{1})(|b|+\bar{1})+(|c|+\bar{1})(|a|+|b|)} c \vdash b \vdash a) \otimes F_{kl}(1), \end{aligned}$$

choosing $c = 1$ we have that

$$F_{ij}(a \dashv b - (-1)^{|a||b|} b \vdash a) \otimes F_{kl}(1) = 0.$$

To see that ψ is a Leibniz superalgebra homomorphism,

$$\begin{aligned} F_{ij}(a) \otimes F_{kl}(b) &= F_{ij}(a) \otimes [F_{ki}(b), F_{il}(1)] = [F_{ij}(a), F_{ki}(b)] \otimes F_{il}(1) \\ &= -(-1)^{|a||b|+|b|} F_{kj}(b \vdash a) \otimes F_{il}(1) \\ &= -(-1)^{|b|} F_{kj}(a \dashv b) \otimes F_{il}(1) \\ &= (-1)^{|b|} F_{ij}(a \dashv b) \otimes F_{kl}(1). \end{aligned}$$

The previous relation also proves that $\psi\circ\varphi$ is the identity map. Moreover, it is straightforward that $\varphi\circ\psi$ is the identity map, completing the proof. □

4.4.4 Case of $\mathfrak{sl}(3,0,D)$

Let $\mathcal{W}(3,0,D)$ be the direct sum of six copies of D_3. The elements will be represented by $v_{ijpq}(a)$ where $pq = ik$ or kj and $\{i,j,k\}=\{1,2,3\}$ and they will be related by R-linearity, the equivalence relations of D_3 and the additional relation $v_{ijpq}(a) = -v_{pqij}(a)$.

Theorem 4.4.6. *The universal central extension of* $\mathfrak{sl}(3,0,D)$ *is* $\mathfrak{stl}(3,0,D)\oplus D_3^6$.

Proof. Let be the homomorphism

$$\varphi\colon \mathfrak{stl}(3,0,D)\otimes\mathfrak{stl}(3,0,D)\to\mathfrak{stl}(3,0,D)\oplus D_3^6,$$

defined on generators by

$$F_{ij}(a)\otimes F_{pq}(b)\mapsto\begin{cases} v_{ijpq}(a\dashv b) & \text{if } pq=ik \text{ or } kj,\\ [F_{ij}(a),F_{pq}(b)] & \text{otherwise.}\end{cases}$$

To check that φ is well defined we only need to check that relation (Gen) is followed when a $v_{ijpq}(a)$ appears. It is immediate that

$$\varphi(x\otimes[y,z]) = -(-1)^{|y||z|}\varphi(x\otimes[z,y]),$$

so the non straightforward cases are

$$\begin{aligned}\varphi(F_{ji}(a)\otimes[F_{ji}(b),F_{ik}(c)]) &= v_{jijk}(a\dashv b\dashv c)\\ &= v_{jijk}\big(a\dashv(b\vdash c)\big) = -(-1)^{|b||c|}v_{jkji}(a\dashv c\dashv b)\\ &= -(-1)^{|b||c|}\varphi\big(F_{jk}(a\dashv c)\otimes F_{ji}(b)\big)\\ &= \varphi\big([F_{ji}(a),F_{ji}(b)]\otimes F_{ik}(c)\\ &\quad -(-1)^{|b||c|}[F_{ji}(a),F_{ik}(c)]\otimes F_{ji}(b)\big),\end{aligned}$$

and

$$\begin{aligned}\varphi(F_{ij}(a)\otimes[F_{ki}(b),F_{ij}(c)]) &= v_{ijkj}(a\dashv b\dashv c)\\ &= -(-1)^{|a||b|}v_{kjij}\big((b\vdash a)\dashv c\big)\\ &= -(-1)^{|a||b|}\varphi\big(F_{kj}(b\vdash a)\otimes F_{ij}(c)\big)\\ &= \varphi\big([F_{ij}(a),F_{ki}(b)]\otimes F_{ij}(c)\\ &\quad -(-1)^{|b||c|}[F_{ij}(a),F_{ij}(c)]\otimes F_{ki}(b)\big).\end{aligned}$$

Now we define $\psi\colon \mathfrak{stl}(3,0,D)\oplus D_3^6\to\mathfrak{stl}(3,0,D)\otimes\mathfrak{stl}(3,0,D)$ by $F_{ij}(a)\mapsto F_{ik}(a)\otimes F_{kj}(1)$ and $v_{ijpq}(a)\mapsto F_{ij}(a)\otimes F_{pq}(1)$. There is only one choice for k, but we need to check that it is well defined for elements of $\mathfrak{stl}(3,0,D)$, since the arguments of Theorem 4.3.1 do not hold.

Then,

$$\begin{aligned}F_{ik}(a\dashv b)\otimes F_{kj}(1) &= -F_{ij}(a\dashv b)\otimes[F_{kj}(1),F_{jk}(1)]\\ &= -[F_{ik}(a),F_{kj}(b)]\otimes[F_{kj}(1),F_{jk}(1)]\\ &= F_{ik}(a)\otimes\big(F_{kj}(b)+F_{kj}(b)\big)-F_{ik}(a)\otimes F_{kj}(b)\\ &= F_{ik}(a)\otimes F_{kj}(b),\end{aligned}$$

and similarly for $F_{ik}(a\dashv b)\otimes F_{ji}(1)=F_{ik}(a)\otimes F_{ji}(b)$. Moreover,

$$F_{ij}(a)\otimes F_{ij}(b)=F_{ij}(a)\otimes[F_{ik}(b),F_{kj}(1)]=0.$$

For the elements of D_3^6,

$$0=[F_{ij}(a),F_{ik}(1)]\otimes[F_{ik}(1),F_{ki}(1)]=F_{ij}(a)\otimes F_{ik}(-3)=F_{ij}(3a)\otimes F_{ik}(1),$$

and

$$\begin{aligned}&0=[F_{ij}(a),F_{ji}(b)]\otimes[F_{ij}(c),F_{ik}(1)]\\ &=F_{ij}(a\dashv b\dashv c+(-1)^{|a||b|+|a||c|+|b||c|}c\vdash b\vdash a)\otimes F_{ik}(1)-F_{ik}(a\dashv b)\otimes F_{ij}(c),\end{aligned}$$

choosing $b=c=1$,

$$F_{ik}(a)\otimes F_{ij}(1)=-F_{ij}(a)\otimes F_{ik}(1),$$

and choosing $c=1$,

$$F_{ij}(-a\dashv b+(-1)^{|a||b|}b\vdash a)\otimes F_{ik}(1)=0.$$

To complete the proof,

$$F_{ij}(a) \otimes F_{ik}(b) = -F_{ij}(a) \otimes [F_{jk}(b), F_{ij}(1)] = -F_{ik}(a \dashv b) \otimes F_{ij}(1)$$
$$= F_{ij}(a \dashv b) \otimes F_{ik}(1).$$

□

4.4.5 Case of $\mathfrak{sl}(2,1,D)$

In this case, $\mathcal{W}(2,1,D) = 0$.

Theorem 4.4.7. *The universal central extension of* $\mathfrak{sl}(2,1,D)$ *is* $\mathfrak{stl}(2,1,D)$*.*

Proof. Defining the homomorphisms as in Theorem 4.4.6, we can recover the relations and additionally

$$0 = [F_{ij}(a), F_{ik}(b)] \otimes [F_{ik}(1), F_{ki}(1)] = F_{ij}(a) \otimes F_{ik}(2 + (-1)^{|i|+|k|}1).$$

Therefore, if $|i| = \bar{1}$ or $|k| = \bar{1}$, we have the relation $F_{ij}(a) \otimes F_{ik}(1) = 0$. If $|j| = \bar{1}$, we do the same calculation but for $F_{ik}(a) \otimes F_{ij}(1)$. It is similar for $F_{ij}(a) \otimes F_{kj}(1)$. □

4.5 Hochschild homology and Leibniz homology

In this section we adapt to the superalgebra case the definition of Hochschild homology of dialgebras introduced in [7] and we relate it with the universal central extension of $\mathfrak{sl}(m,n,D)$. From now on, we assume that D is R-free.

Let D be a superdialgebra with an R-basis containing the bar-unit. The boundary map $d_n \colon D^{\otimes n+1} \to D^{\otimes n}$ is defined on generators by

$$d_n(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (a_0 \otimes \cdots \otimes a_i \dashv a_{i+1} \otimes \cdots \otimes a_n)$$
$$+ (-1)^{n+|a_n| \sum_{i=0}^{n-1} |a_i|}(a_n \vdash a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}),$$

where $a_i \in D$. The *Hochschild homology of superdialgebras*, denoted by $\mathrm{HH}_*(D)$, is the homology of the chain complex formed by the boundary maps d_*. Let I be the ideal of D generated by the elements of the form $a \otimes b \dashv c - a \otimes b \vdash c$. We define

$$\mathrm{HHS}_1(D) = \frac{\mathsf{Ker}\, d_1}{\mathrm{Im}\, d_2 + I}.$$

Theorem 4.5.1. *There is an isomorphism of R-supermodules* $\mathrm{HL}_2\big(\mathfrak{sl}(m,n,D)\big) \cong \mathrm{HHS}_1(D) \oplus \mathcal{W}(m,n,D)$.

Proof. We have the following diagram

$$\begin{array}{ccccccc}
0 \to \mathrm{HHS}_1(D)\oplus\mathcal{W}(m,n,D) & \longrightarrow & \dfrac{D\otimes D}{\operatorname{Im} d_2 + I}\oplus\mathcal{W}(m,n,D) & \xrightarrow{d_1} & [D,D] & \longrightarrow & 0\\
 & & \mathrm{Str}_2 \Big\uparrow\Big\downarrow \mu & & \mathrm{Str}_1 \Big\uparrow\Big\downarrow E_{11}(-) & & \\
0 \longrightarrow \mathrm{HL}_2\big(\mathfrak{sl}(m,n,D)\big) & \longrightarrow & \mathfrak{stl}(m,n,D)\otimes\mathfrak{stl}(m,n,D) & \xrightarrow{\omega} & \mathfrak{sl}(m,n,D) & \to & 0,
\end{array}$$

where $\mu(a\otimes b) = F_{1j}(a)\otimes F_{j1}(b) - (-1)^{|a||b|}F_{1j}(b \vdash a)\otimes F_{j1}(1)$, $\mu\big(v_{ijkl}(a)\big) = F_{ij}(a)\otimes F_{kl}(b)$ and

$$\mathrm{Str}_2\big(F_{ij}(a)\otimes F_{kl}(b)\big)\begin{cases} a\otimes b & \text{if } i=j \text{ and } k=l\\ v_{ijkl}(a\dashv b) & \text{if it makes sense depending on } m,n\\ 0 & \text{otherwise.}\end{cases}$$

It is a straightforward computation that $\mu\circ\mathrm{Str}_2$ and $E_{11}(-)\circ\mathrm{Str}_1$ are the identity maps and that the diagram is commutative. Then the restriction of Str_2 to the kernel of ω is also a split epimorphism, with μ restricted to the kernel of d_1 as section. Let us see that these restrictions are indeed isomorphisms. An element in the kernel of ω is a sum of elements of the form $F_{ij}(a)\otimes F_{ji}(b)$ plus elements of $\mathcal{W}(m,n,D)$. Any element of $\mathsf{Ker}\,\omega$ can be written as an element of $\mathsf{Im}\,\mu$ plus $\sum_{i=2}^{m+n} F_{1i}(a_i)\otimes F_{i1}(1)$, since

$$F_{ij}(a)\otimes F_{ji}(b) = F_{i1}(a)\otimes F_{1i}(b) - (-1)^{(|a|+|i|+|j|)(|b|+|i|+|j|)}F_{j1}(b\vdash a)\otimes F_{1j}(1),$$

and

$$\begin{aligned} F_{1j}(a)\otimes F_{j1}(b) = F_{1j}(a)\otimes F_{j1}(b) - (-1)^{|a||b|}F_{j1}(b\vdash a)\otimes F_{1j}(1) +{}&\\ (-1)^{|a||b|}F_{j1}(b\vdash a)\otimes F_{1j}(1).&\end{aligned}$$

Furthermore, if it is in the kernel of ω, all the a_i must be zero. Then the restriction of μ to the kernel of d_1 is surjective. □

Remark 4.5.2. The proof given in [21] can also be adapted since the assumptions on the characteristic of the ring are not used, but we rather give our version of the proof to show its relation with non-abelian tensor product.

4.6 Concluding remarks

Combining the results obtained above we present the following summarizing theorems

Theorem 4.6.1. *Let R a unital commutative ring and D an associative unital R-superdialgebra with an R-basis containing the bar-unit. Then,*

$$\mathrm{HL}_2\left(\mathfrak{sl}(m,n,D)\right) = \begin{cases} \mathrm{HHS}_1(D) & \text{for } m+n \geqslant 5 \text{ or } m=2, n=1, \\ \mathrm{HHS}_1(D) \oplus D_3^6 & \text{for } m=3, n=0, \\ \mathrm{HHS}_1(D) \oplus D_2^6 & \text{for } m=4, n=0, \\ \mathrm{HHS}_1(D) \oplus \Pi(D_2)^6 & \text{for } m=3, n=1, \\ \mathrm{HHS}_1(D) \oplus D_2^4 \oplus D_0^2 & \text{for } m=2, n=2, \end{cases}$$

where D_m is the quotient of D by the ideal $mD + ([D,D] \dashv D)$ (Definition 4.4.2) and Π is the parity change functor.

Theorem 4.6.2. *Let R a unital commutative ring and D an associative unital R-superdialgebra with an R-basis containing the bar-unit. Then,*

$$\mathrm{HL}_2\left(\mathfrak{stl}(m,n,D)\right) = \begin{cases} 0 & \text{for } m+n \geqslant 5 \text{ or } m=2, n=1, \\ D_3^6 & \text{for } m=3, n=0, \\ D_2^6 & \text{for } m=4, n=0, \\ \Pi(D_2)^6 & \text{for } m=3, n=1, \\ D_2^4 \oplus D_0^2 & \text{for } m=2, n=2, \end{cases}$$

where D_m is the quotient of D by the ideal $mD + ([D,D] \dashv D)$ (Definition 4.4.2) and Π is the parity change functor.

Remark 4.6.3. We recall that in the case that $m = n = 2$, $\mathcal{W}(2,2,D)$ might not be zero even if $\mathrm{char}(R) \neq 2$ contradicting [18, Theorem 6.2].

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Chapter 5

A non-abelian exterior product and homology of Leibniz algebras

Abstract

We introduce a non-abelian exterior product of two crossed modules of Leibniz algebras and investigate its relation to the low-dimensional Leibniz homology. Later this non-abelian exterior product is applied to the construction of an eight term exact sequence in Leibniz homology. Also its relationship to the universal quadratic functor is established, which is applied to the comparison of the second Lie and Leibniz homologies of a Lie algebra.

Reference

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5.1 Introduction

Leibniz algebras were first defined in 1965 by Bloh [2] as a non skew-symmetric analogue of Lie algebras but they became very popular when in 1993 Loday rediscovered them in [22]. One of the main reasons that Loday had to introduce them was that in the Lie homology complex the only property of the bracket needed was the so called Leibniz identity. Therefore, one can think about this notion as "non-commutative" analog of Lie algebras and study its homology. Since then, many authors have been studying them obtaining very relevant algebraic results ([24], [25]) and due their relations with Physics ([20], [26]) and Geometry ([17]). Many results of Lie algebras have been extended to the Leibniz case. As an example of these generalizations, Gnedbaye [18] extended to Leibniz algebras the notion of non-abelian tensor product, defined by Brown and Loday in the context of groups [5] and by Ellis in the context of Lie algebras [15]. The non-abelian tensor product was firstly introduced as a tool in homotopy theory, but it can give us nice information about central extensions and (co)homology.

The notion of non-abelian exterior product was introduced in groups by Brown and Loday [4] and also extended to the Lie case by Ellis [15, 14]. The main objective of this manuscript is to give a proper generalization of this concept to Leibniz algebras. Given two ideals $\mathfrak{a}$ and $\mathfrak{b}$ of a Lie algebra $\mathfrak{g}$, the non-abelian exterior product is the quotient of the non-abelian tensor product $\mathfrak{a} \star \mathfrak{b}$ by the elements of the form $c * c$, where $c \in \mathfrak{a} \cap \mathfrak{b}$. This makes sense in Lie theory since these are elements of the kernel of the homomorphism $\mathfrak{g} \star \mathfrak{g} \to \mathfrak{g}$, $g * g' \mapsto [g, g']$ for all $g, g' \in \mathfrak{g}$, but this is not true in Leibniz algebras due to the lack of antisymmetry. Nevertheless, the non-abelian tensor product of two ideals of a Leibniz algebra, has some duplicity in the elements of the intersection, so this will be the way we can affront the problem. In fact, our definition of the non-abelian exterior product is given for crossed modules of Leibniz algebras, which is more general concept than ideals of Leibniz algebras.

The paper is organized as follows. In Section 5.2 we recall some basic definitions and properties of Leibniz algebras and Leibniz homology. In Section 5.3 we introduce the non-abelian exterior product and we study its connections with the second Leibniz homology. In Section 5.4 we obtain an eight term exact sequence using the non-abelian exterior product. In Section 5.5 we explore the relations with Whitehead's universal quadratic functor. Finally, in Section 5.6 we compare the second Leibniz homology and the second Lie homology of a Lie algebra.

5.2 Leibniz algebras and homology

Throughout the paper $\mathbb{K}$ is a field, unless otherwise stated. All vector spaces and algebras are considered over $\mathbb{K}$, all linear maps are $\mathbb{K}$-linear maps and $\otimes$ stands for $\otimes_{\mathbb{K}}$.

Definition 5.2.1 ([22])**.** A Leibniz algebra is a vector space $\mathfrak{g}$ equipped with a linear map (Leibniz bracket)

$$[\ ,\ ]\colon \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$$

satisfying the Leibniz identity

$$[x,[y,z]] = [[x,y],z] - [[x,z],y],$$

for all $x, y, z \in \mathfrak{g}$.

A homomorphism of Leibniz algebras is a linear map preserving the bracket. The respective category of Leibniz algebras will be denoted by **Lb**.

A subspace $\mathfrak{a}$ of a Leibniz algebra $\mathfrak{g}$ is called (two-sided) *ideal* of $\mathfrak{g}$ if $[a,x]$, $[x,a] \in \mathfrak{a}$ for all $a \in \mathfrak{a}$ and $x \in \mathfrak{g}$. In this case the quotient space $\mathfrak{g}/\mathfrak{a}$ naturally inherits a Leibniz algebra structure.

An example of ideal of a Leibniz algebra $\mathfrak{g}$ is the *commutator* of $\mathfrak{g}$, denoted by $[\mathfrak{g},\mathfrak{g}]$, which is the subspace of $\mathfrak{g}$ spanned by elements of the form $[x,y]$, $x, y \in \mathfrak{g}$. The quotient $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ is denoted by $\mathfrak{g}^{\mathrm{ab}}$ and is called *abelianization* of $\mathfrak{g}$. One more example of an ideal is the *center* $C(\mathfrak{g}) = \{c \in \mathfrak{g} \mid [x,c] = 0 = [c,x],\ \text{for all } x \in \mathfrak{g}\}$ of $\mathfrak{g}$. Note that both $\mathfrak{g}^{\mathrm{ab}}$ and $C(\mathfrak{g})$ are *abelian Leibniz algebras*, that is, Leibniz algebras with the trivial Leibniz bracket $[\ ,\ ] = 0$.

Clearly, any Lie algebra is a Leibniz algebra and conversely, any Leibniz algebra with the antisymmetric Leibniz bracket is a Lie algebra. This is why Leibniz algebras are called non-commutative generalization of Lie algebras. Thus, there is a full embedding functor **Lie** $\hookrightarrow$ **Lb**, where **Lie** denotes the category of Lie algebras. This embedding has a left adjoint $\mathfrak{Lie}\colon$ **Lb** $\to$ **Lie**, called the Liezation functor and defined as follows. Given a Leibniz algebra $\mathfrak{g}$, $\mathfrak{Lie}(\mathfrak{g})$ is the quotient of $\mathfrak{g}$ by the subspace (that automatically is an ideal) spanned by elements of the form $[x,x]$, $x \in \mathfrak{g}$ (see for example [21]).

The original reason to introduce Leibniz algebras was a new variant of Lie homology, called non-commutative Lie homology or Leibniz homology, developed in [23, 24] and denoted by HL_*. Let us recall the definition of HL_*.

Given a Leibniz algebra $\mathfrak{g}$, consider the following chain complex:

$$CL_*(\mathfrak{g}) \equiv \quad \cdots \xrightarrow{d} \mathfrak{g}^{\otimes n} \xrightarrow{d} \mathfrak{g}^{\otimes n-1} \xrightarrow{d} \cdots \xrightarrow{d} \mathfrak{g} \xrightarrow{d} \mathbb{K},$$

where the boundary map d is given by

$$\begin{aligned}
&d(x) = 0, \ \text{ for each } \ x \in \mathfrak{g};\\
&d(x_1 \otimes \cdots \otimes x_n)\\
&= \sum_{1 \leqslant i < j \leqslant n} (-1)^j (x_1 \otimes \cdots \otimes x_{i-1} \otimes [x_i, x_j] \otimes x_{i+1} \otimes \cdots \otimes \widehat{x_j} \otimes \cdots \otimes x_n),
\end{aligned}$$

for $x_1, \ldots, x_n \in \mathfrak{g}$ and $n > 1$. The n-th homology group of $\mathfrak{g}$ is defined by

$$HL_n(\mathfrak{g}) = H_n(CL_*(\mathfrak{g})), \ \ n \geqslant 0.$$

5.2.1 Homology via derived functors

Let X be a set and M_X be the free magma on X with a binary operation $[\ ,\ ]$. Denote by $\mathbb{K}(M_X)$ the free vector space over the set M_X. In a natural way we can extend $[\ ,\ ]$ to a binary operation $[\ ,\ ]$ on $\mathbb{K}(M_X)$. Thus $\mathbb{K}(M_X)$ is the free algebra on X (see e.g. [28]). For any subset $S \subseteq \mathbb{K}(M_X)$, let $N^1(S)$ denote the vector subspace of $\mathbb{K}(M_X)$ generated by S and $\{[x,s],[s,x] \mid x \in M_X, s \in S\}$. Let $\mathcal{N}(S) = \underset{i \geqslant 1}{\cup} N^i(S)$ where for $i > 1$, $N^i(S) = N^1(N^{i-1}(S))$. In particular, consider $S_X = \{[x,[y,z]] - [[x,y],z] + [[x,z],y] \mid x,y,z \in M_X\}$ and denote $\mathbb{K}(M_X)/\mathcal{N}(S_X)$ by $\mathfrak{F}(X)$. In other words, $\mathfrak{F}(X)$ is the quotient of $\mathbb{K}(M_X)$ by the two-sided ideal generated by the subset S_X. Clearly $\mathfrak{F}(X)$ is a Leibniz algebra, called the free Leibniz algebra on the set X. It is easy to see that the construction $X \mapsto \mathfrak{F}(X)$ defines a covariant functor

$$\mathfrak{F} \colon \textbf{Set} \to \textbf{Lb},$$

which is left adjoint to the natural forgetful functor $\mathfrak{U} \colon \textbf{Lb} \to \textbf{Set}$, where **Set** denote the category of sets.

Remark 5.2.2. Let **Vect** denote the category of vector spaces. There is a functor $\mathfrak{F}_1 \colon \textbf{Vect} \to \textbf{Lb}$ assigning to each vector space V the free Leibniz algebra over V (see [21]). Then $\mathfrak{F} = \mathfrak{F}_1 \circ \mathfrak{F}_2$, where $\mathfrak{F}_2 \colon \textbf{Set} \to \textbf{Vect}$ associates to each set X the vector space with basis X.

It is well known that every adjoint pair of functors induces a cotriple (see for example [1, Chapter 1] or [19, Chapter 2]). Let $\mathbb{F} = (\mathbb{F}, \delta, \tau)$ denote the cotriple in $\mathbf{Lb}$ defined by the adjunction $\mathfrak{F} \dashv \mathfrak{U}$, that is, $\mathbb{F} = \mathfrak{F}\mathfrak{U}\colon \mathbf{Lb} \to \mathbf{Lb}$, $\tau\colon \mathbb{F} \to 1_{\mathbf{Lb}}$ is the counit and $\delta = \mathfrak{F}u\mathfrak{U}\colon \mathbb{F} \to \mathbb{F}^2$, where $u\colon 1_{\mathbf{Set}} \to \mathfrak{U}\mathfrak{F}$ is the unit of the adjunction. Then, given an endofunctor $\mathfrak{T}\colon \mathbf{Lb} \to \mathbf{Lb}$, one can consider derived functors $\mathcal{L}_n^{\mathbb{F}}\mathfrak{T}\colon \mathbf{Lb} \to \mathbf{Lb}$, $n \geqslant 0$, with respect to the cotriple $\mathbb{F}$ (see again [1]). In particular, Leibniz homology can be described in terms of the derived functors of the abelianization functor $\mathfrak{Ab}\colon \mathbf{Lb} \to \mathbf{Lb}$, defined by $\mathfrak{Ab}(\mathfrak{g}) = \mathfrak{g}^{\mathrm{ab}}$.

Theorem 5.2.3. *Let $\mathfrak{g}$ be a Leibniz algebra. Then there is an isomorphism*

$$HL_{n+1}(\mathfrak{g}) \cong \mathcal{L}_n^{\mathbb{F}}\mathfrak{Ab}(\mathfrak{g}), \quad n \geqslant 0.$$

Proof. Let $\mathfrak{f}_*$ be the $\mathbb{F}$-cotriple simplicial resolution of $\mathfrak{g}$, that is, $\mathfrak{f}_* = (\mathfrak{f}_*(\mathfrak{g}), d_i^n, s_i^n)$ is the simplicial Leibniz algebra with

$$\begin{aligned} &\mathfrak{f}_n(\mathfrak{g}) = \mathbb{F}^{n+1}(\mathfrak{g}) = \mathbb{F}\big(\mathbb{F}^n(\mathfrak{g})\big), \\ &d_i^n = \mathbb{F}^i(\tau_{\mathbb{F}^{n-i}}), \quad s_i^n = \mathbb{F}^i(\delta_{\mathbb{F}^{n-i}}), \quad 0 \leqslant i \leqslant n, \quad n \geqslant 0. \end{aligned}$$

Applying the functor CL_* to $\mathfrak{f}_*$ dimension-wise, we get the following bicomplex:

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
{\scriptstyle d}\downarrow & & {\scriptstyle -d}\downarrow & & {\scriptstyle d}\downarrow & & \\
\mathfrak{f}_0^{\otimes 2} & \longleftarrow & \mathfrak{f}_1^{\otimes 2} & \longleftarrow & \mathfrak{f}_2^{\otimes 2} & \longleftarrow & \cdots \\
{\scriptstyle d}\downarrow & & {\scriptstyle -d}\downarrow & & {\scriptstyle d}\downarrow & & \\
\mathfrak{f}_0 & \longleftarrow & \mathfrak{f}_1 & \longleftarrow & \mathfrak{f}_2 & \longleftarrow & \cdots \\
{\scriptstyle d}\downarrow & & {\scriptstyle -d}\downarrow & & {\scriptstyle d}\downarrow & & \\
\mathbb{K} & \longleftarrow & \mathbb{K} & \longleftarrow & \mathbb{K} & \longleftarrow & \cdots,
\end{array}$$

where the horizontal differentials are obtained by alternating sums of face homomorphisms. Since $\mathbb{K}$ is a field, $\mathfrak{f}_*^{\otimes n} \to \mathfrak{g}^{\otimes n}$ is an aspherical augmented simplicial vector space (see for example [10, Lemma 2.3] or [11, Lemma 2.1]), for each $n \geqslant 1$. Therefore, we have an isomorphism:

$$HL_n(\mathfrak{g}) \cong H_n(Tot(CL_*(\mathfrak{f}_*))), \quad n \geqslant 0.$$

On the other hand we have the following spectral sequence:

$$E^1_{pq} = H_q(\mathfrak{f}_p) \Rightarrow H_{p+q}(Tot(CL_*(\mathfrak{f}_*))).$$

Since $\mathfrak{f}_n$ is a free Leibniz algebra for each $n \geqslant 0$, we obtain:

$$E^1_{pq} = \begin{cases} \mathbb{K} & \text{for} \quad q = 0 \ , \\ \mathfrak{f}_p/[\mathfrak{f}_p, \mathfrak{f}_p] & \text{for} \quad q = 1 \ , \\ 0 & \text{for} \quad q > 1 \ . \end{cases}$$

This spectral sequence degenerates at E^2 and

$$E^2_{pq} = \begin{cases} \mathbb{K} & \text{for} \quad q = 0 \quad \text{and} \quad p = 0 \ , \\ 0 & \text{for} \quad q = 0 \quad \text{and} \quad p > 0 \ , \\ H_p(\mathfrak{Ab}(\mathfrak{f}_*)) & \text{for} \quad q = 1 \ , \\ 0 & \text{for} \quad q > 1 \ . \end{cases}$$

Thus, the spectral sequence argument completes the proof. □

Remark 5.2.4. The $\mathbb{F}$-cotriple simplicial resolution of a Leibniz algebra $\mathfrak{g}$ is a free (projective) simplicial resolution of $\mathfrak{g}$ and by [1, 5.3], if $\mathfrak{f}_*$ is any of them, then there are natural isomorphisms

$$HL_n(\mathfrak{g}) \cong \pi_{n-1}(\mathfrak{Ab}(\mathfrak{f}_*)), \quad n \geqslant 1.$$

5.2.2 Hopf formulas

Theorem 5.2.3 enables us to prove Hopf formulas for the (higher) homology of Leibniz algebras, pursuing the line and technique developed in [13] for the description of higher group homology via Hopf formulas (see [3]). In this respect herewith we state two theorems without proofs. In fact they are particular cases of [7, Theorem 15] describing homology of Leibniz n-algebras via Hopf formulas. Note also that, of course, these results agree with the categorical approach to the problem given in [16] for semi-abelian homology.

An extension of Leibniz algebras $0 \to \mathfrak{r} \to \mathfrak{f} \to \mathfrak{g} \to 0$ is said to be a free presentation of $\mathfrak{g}$, if $\mathfrak{f}$ is a free Leibniz algebra over a set.

Theorem 5.2.5. *Let $0 \to \mathfrak{r} \to \mathfrak{f} \to \mathfrak{g} \to 0$ be a free presentation of a Leibniz algebra $\mathfrak{g}$. Then there is an isomorphism:*

$$HL_2(\mathfrak{g}) \cong \frac{\mathfrak{r} \cap [\mathfrak{f}, \mathfrak{f}]}{[\mathfrak{r}, \mathfrak{f}]}.$$

Theorem 5.2.6. *Let* $\mathfrak{t}$ *and* $\mathfrak{s}$ *be ideals of a free Leibniz algebra* $\mathfrak{f}$. *Suppose that* $\mathfrak{f}/\mathfrak{t}$ *and* $\mathfrak{f}/\mathfrak{s}$ *are free Leibniz algebras, and that* $\mathfrak{g} = \mathfrak{f}/(\mathfrak{t}+\mathfrak{s})$. *Then there is an isomorphism:*

$$HL_3(\mathfrak{g}) \cong \frac{\mathfrak{t} \cap \mathfrak{s} \cap [\mathfrak{f},\mathfrak{f}]}{[\mathfrak{t},\mathfrak{s}] + [\mathfrak{t} \cap \mathfrak{s}, \mathfrak{f}]}.$$

5.3 Non-abelian tensor and exterior product of Leibniz algebras

5.3.1 Leibniz actions and crossed modules

Definition 5.3.1. Let $\mathfrak{m}$ and $\mathfrak{n}$ be Leibniz algebras. A *Leibniz action* of $\mathfrak{m}$ on $\mathfrak{n}$ is a couple of bilinear maps $\mathfrak{m} \times \mathfrak{n} \to \mathfrak{n}$, $(m,n) \mapsto {}^{m}n$, and $\mathfrak{n} \times \mathfrak{m} \to \mathfrak{n}$, $(n,m) \mapsto n^{m}$, satisfying the following axioms:

$$\begin{aligned}
{}^{[m,m']}n &= {}^{m}({}^{m'}n) + ({}^{m}n)^{m'}, & {}^{m}[n,n'] &= [{}^{m}n,n'] - [{}^{m}n',n],\\
n^{[m,m']} &= (n^{m})^{m'} - (n^{m'})^{m}, & [n,n']^{m} &= [n^{m},n'] + [n,n'^{m}],\\
{}^{m}({}^{m'}n) &= -{}^{m}(n^{m'}), & [n,{}^{m}n'] &= -[n,n'^{m}],
\end{aligned}$$

for each $m,m' \in \mathfrak{m}$, $n,n' \in \mathfrak{n}$. For example, if $\mathfrak{m}$ is a subalgebra of a Leibniz algebra $\mathfrak{g}$ (maybe $\mathfrak{g} = \mathfrak{m}$) and $\mathfrak{n}$ is an ideal of $\mathfrak{g}$, then the Leibniz bracket in $\mathfrak{g}$ yields a Leibniz action of $\mathfrak{m}$ on $\mathfrak{n}$.

Definition 5.3.2. A *Leibniz crossed module* $(\mathfrak{m},\mathfrak{g},\eta)$ is a homomorphism of Leibniz algebras $\eta\colon \mathfrak{m} \to \mathfrak{g}$ together with an action of $\mathfrak{g}$ on $\mathfrak{m}$ such that

$$\begin{aligned}
&\eta({}^{x}m) = [x,\eta(m)], \qquad \eta(m^{x}) = [\eta(m),x],\\
&{}^{\eta(m_1)}m_2 = [m_1,m_2] = m_1^{\eta(m_2)},
\end{aligned}$$

where $x \in \mathfrak{g}$ and $m,m_1,m_2 \in \mathfrak{m}$.

Example 5.3.3. Let $\mathfrak{a}$ be an ideal of a Leibniz algebra $\mathfrak{g}$. Then the inclusion $i\colon \mathfrak{a} \to \mathfrak{g}$ is a crossed module where the action of $\mathfrak{g}$ on $\mathfrak{a}$ is given by the bracket in $\mathfrak{g}$. In particular, a Leibniz algebra may be regarded as a crossed module in two ways, as the inclusion $i\colon 0 \to \mathfrak{g}$ and as the identity map $1_{\mathfrak{g}}\colon \mathfrak{g} \to \mathfrak{g}$.

5.3.2 Non-abelian tensor product

Let $\mathfrak{m}$ and $\mathfrak{n}$ be Leibniz algebras with mutual actions on one another. The *non-abelian tensor product* of $\mathfrak{m}$ and $\mathfrak{n}$, denoted by $\mathfrak{m} \star \mathfrak{n}$, is defined in [18] to be the Leibniz algebra generated by the symbols $m * n$ and $n * m$, for all $m \in \mathfrak{m}$ and $n \in \mathfrak{n}$, subject to the following relations:

$$
\begin{aligned}
&(1a) && k(m*n) = km*n = m*kn,\\
&(1b) && k(n*m) = kn*m = n*km,\\
&(2a) && (m+m')*n = m*n + m'*n,\\
&(2b) && (n+n')*m = n*m + n'*m,\\
&(2c) && m*(n+n') = m*n + m*n',\\
&(2d) && n*(m+m') = n*m + n*m',\\
&(3a) && m*[n,n'] = m^n * n' - m^{n'} * n,\\
&(3b) && n*[m,m'] = n^m * m' - n^{m'} * m,\\
&(3c) && [m,m']*n = {}^m n * m' - m * n^{m'},\\
&(3d) && [n,n']*m = {}^n m * n' - n * m^{n'},\\
&(4a) && m * {}^{m'}n = -m * n^{m'},\\
&(4b) && n * {}^{n'}m = -n * m^{n'},\\
&(5a) && m^n * {}^{m'}n' = [m*n, m'*n'] = {}^m n * m'^{n'},\\
&(5b) && {}^n m * n'^{m'} = [n*m, n'*m'] = n^m * {}^{n'}m',\\
&(5c) && m^n * n'^{m'} = [m*n, n'*m'] = {}^m n * {}^{n'}m',\\
&(5d) && {}^n m * {}^{m'}n' = [n*m, m'*n'] = n^m * m'^{n'},
\end{aligned}
$$

for each $k \in \mathbb{K}$, $m, m' \in \mathfrak{m}$ and $n, n' \in \mathfrak{n}$.

There are induced homomorphisms of Leibniz algebras $\tau_{\mathfrak{m}} \colon \mathfrak{m} \star \mathfrak{n} \to \mathfrak{m}$ and $\tau_{\mathfrak{n}} \colon \mathfrak{m} \star \mathfrak{n} \to \mathfrak{n}$ where $\tau_{\mathfrak{m}}(m*n) = m^n$, $\tau_{\mathfrak{m}}(n*m) = {}^n m$, $\tau_{\mathfrak{n}}(m*n) = {}^m n$ and $\tau_{\mathfrak{n}}(n*m) = n^m$. Note that, in the case of compatible actions (see [18] for the definition), both $\tau_{\mathfrak{m}}$ and $\tau_{\mathfrak{m}}$ are crossed modules of Leibniz algebras.

5.3.3 Non-abelian exterior product

Let us consider two Leibniz crossed modules $\eta \colon \mathfrak{m} \to \mathfrak{g}$ and $\mu \colon \mathfrak{n} \to \mathfrak{g}$. Then there are induced actions of $\mathfrak{m}$ and $\mathfrak{n}$ on each other via the action of $\mathfrak{g}$. Therefore, we can consider the non-abelian tensor product $\mathfrak{m} \star \mathfrak{n}$. We define $\mathfrak{m} \square \mathfrak{n}$ as the vector subspace of $\mathfrak{m} \star \mathfrak{n}$ generated by the elements $m * n' - n * m'$ such that $\eta(m) = \mu(n)$ and $\eta(m') = \mu(n')$.

Proposition 5.3.4. *The vector subspace* $\mathfrak{m} \square \mathfrak{n}$ *is contained in the center of* $\mathfrak{m} \star \mathfrak{n}$*, so in particular it is an ideal of* $\mathfrak{m} \star \mathfrak{n}$*.*

Proof. Everything can be readily checked by using defining relations (5a)-(5d) of $\mathfrak{m} \star \mathfrak{n}$. For instance, for any $m'' \in \mathfrak{m}$ and $n'' \in \mathfrak{n}$, we have

$$\begin{aligned}
[m * n' - n * m', m'' * n''] &= m^{n'} * {}^{m''}n'' - {}^{n}m' * {}^{m''}n'' \\
&= m^{\mu(n')} * {}^{m''}n'' - {}^{\mu(n)}m' * {}^{m''}n'' \\
&= m^{\eta(m')} * {}^{m''}n'' - {}^{\eta(m)}m' * {}^{m''}n'' \\
&= [m, m'] * {}^{m''}n'' - [m, m'] * {}^{m''}n'' \\
&= 0.
\end{aligned}$$

□

Definition 5.3.5. Let $\eta \colon \mathfrak{m} \to \mathfrak{g}$ and $\mu \colon \mathfrak{n} \to \mathfrak{g}$ be two Leibniz crossed modules in the previous setting. We define the *non-abelian exterior product* $\mathfrak{m} \curlywedge \mathfrak{n}$ of $\mathfrak{m}$ and $\mathfrak{n}$ by

$$\mathfrak{m} \curlywedge \mathfrak{n} = \frac{\mathfrak{m} \star \mathfrak{n}}{\mathfrak{m} \square \mathfrak{n}}.$$

The cosets of $m*n$ and $n*m$ will be denoted by $m \curlywedge n$ and $n \curlywedge m$, respectively.

Remark 5.3.6. Definitions of the non-abelian tensor and exterior products do not require $\mathbb{K}$ to be necessarily a field. It is clear that one can do the same for a commutative ring with identity.

There is an epimorphism of Leibniz algebras $\pi \colon \mathfrak{m} \star \mathfrak{n} \to \mathfrak{m} \curlywedge \mathfrak{n}$ sending $m * n$ and $n * m$ to $m \curlywedge n$ and $n \curlywedge m$, respectively.

To avoid any confusion, let us note that, given a Leibniz algebra $\mathfrak{g}$, for each $x, y \in \mathfrak{g}$, the non-abelian tensor square $\mathfrak{g} \star \mathfrak{g}$ has two copies of generators of the form $x * y$, and exactly these generators are identified in the non-abelian exterior square $\mathfrak{g} \curlywedge \mathfrak{g}$. Thus we need to distinguish two copies of inclusions (identity maps) $i_1, i_2 \colon \mathfrak{g} \to \mathfrak{g}$, $i_i = i_2 = 1_{\mathfrak{g}}$ and take $\mathfrak{g} \curlywedge \mathfrak{g}$ to be the quotient of $\mathfrak{g} \star \mathfrak{g}$ by the relation $i_1(x) * i_2(y) = i_2(x) * i_1(y)$ for each $x, y \in \mathfrak{g}$.

In the case of $\mathfrak{a}$ and $\mathfrak{b}$ being two ideals of a Leibniz algebra $\mathfrak{g}$ seen as crossed modules, the non-abelian exterior product $\mathfrak{a} \curlywedge \mathfrak{b}$ is just $\mathfrak{a} \star \mathfrak{b}$ quotient by the elements of the form $i_1(c) \curlywedge i_2(c') - i_2(c) \curlywedge i_1(c')$, where $c, c' \in \mathfrak{a} \cap \mathfrak{b}$; $i_1 \colon \mathfrak{a} \cap \mathfrak{b} \to \mathfrak{a}$ and $i_2 \colon \mathfrak{a} \cap \mathfrak{b} \to \mathfrak{b}$ are the natural inclusions.

The proof of the following proposition is immediate.

Proposition 5.3.7. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of a Leibniz algebra. There is a homomorphism of Leibniz algebras*

$$\theta_{\mathfrak{a},\mathfrak{b}} \colon \mathfrak{a} \curlywedge \mathfrak{b} \to \mathfrak{a} \cap \mathfrak{b}$$

defined on generators by $\theta_{\mathfrak{a},\mathfrak{b}}(a \curlywedge b) = [a, b]$ and $\theta_{\mathfrak{a},\mathfrak{b}}(b \curlywedge a) = [b, a]$, for all $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$. Moreover, $\theta_{\mathfrak{a},\mathfrak{b}}$ is a crossed module of Leibniz algebras (c.f. [18, Proposition 4.3]).

Proposition 5.3.8. *Let $\mathfrak{g}$ be a perfect Leibniz algebra, that is, $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$. Then $\mathfrak{g} \star \mathfrak{g} = \mathfrak{g} \curlywedge \mathfrak{g}$ and the homomorphism $\theta_{\mathfrak{g},\mathfrak{g}} \colon \mathfrak{g} \curlywedge \mathfrak{g} \twoheadrightarrow \mathfrak{g}$ is the universal central extension of $\mathfrak{g}$.*

Proof. The last four identities of the non-abelian tensor product immediately imply that $\mathfrak{g} \star \mathfrak{g} = \mathfrak{g} \curlywedge \mathfrak{g}$. Hence, by [18, Theorem 6.5] the homomorphism $\theta_{\mathfrak{g},\mathfrak{g}} \colon \mathfrak{g} \curlywedge \mathfrak{g} \twoheadrightarrow \mathfrak{g}$ is the universal central extension of the perfect Leibniz algebra $\mathfrak{g}$. □

5.3.4 Relationship to the second homology

Let $\gamma \colon \mathfrak{g} \to \mathfrak{h}$ be a homomorphism of Leibniz algebras, $\mathfrak{a}$ and $\mathfrak{a}'$ (resp. $\mathfrak{b}$ and $\mathfrak{b}'$) be two ideals of $\mathfrak{g}$ (resp. $\mathfrak{h}$) such that $\gamma(\mathfrak{a}) \subseteq \mathfrak{b}$ and $\gamma(\mathfrak{a}') \subseteq \mathfrak{b}'$. Since $\gamma(\mathfrak{a} \cap \mathfrak{a}') \subseteq \mathfrak{b} \cap \mathfrak{b}'$, it is easy to see that γ induces a homomorphism of Leibniz algebras $\mathfrak{a} \curlywedge \mathfrak{a}' \to \mathfrak{b} \curlywedge \mathfrak{b}'$ in the natural way: $a \curlywedge a' \mapsto \gamma(a) \curlywedge \gamma(a')$ and $a' \curlywedge a \mapsto \gamma(a') \curlywedge \gamma(a)$, for all $a \in \mathfrak{a}$, $a' \in \mathfrak{a}'$.

Now suppose that $0 \to \mathfrak{a} \to \mathfrak{g} \to \mathfrak{h} \to 0$ and $0 \to \mathfrak{a}' \to \mathfrak{g}' \to \mathfrak{h}' \to 0$ are extensions of Leibniz algebras, where $\mathfrak{a}'$ and $\mathfrak{g}'$ are ideals of $\mathfrak{g}$, while $\mathfrak{h}'$ is an ideal of $\mathfrak{h}$. Then the following naturally induced map $\mathfrak{g} \curlywedge \mathfrak{a}' \times \mathfrak{a} \curlywedge \mathfrak{g}' \to \mathfrak{g} \curlywedge \mathfrak{g}'$ is not in general a homomorphism of Leibniz algebras, but the following sequence

$$\mathfrak{g} \curlywedge \mathfrak{a}' \times \mathfrak{a} \curlywedge \mathfrak{g}' \to \mathfrak{g} \curlywedge \mathfrak{g}' \to \mathfrak{h} \curlywedge \mathfrak{h}' \to 0 \tag{5.3.1}$$

is exact, in the sense that $\mathsf{Im}\left(\mathfrak{g} \curlywedge \mathfrak{a}' \times \mathfrak{a} \curlywedge \mathfrak{g}' \to \mathfrak{g} \curlywedge \mathfrak{g}'\right) = \mathsf{Ker}\left(\mathfrak{g} \curlywedge \mathfrak{g}' \to \mathfrak{h} \curlywedge \mathfrak{h}'\right)$.

Lemma 5.3.9. *Let $0 \to \mathfrak{a} \to \mathfrak{g} \to \mathfrak{h} \to 0$ be an extension of Leibniz algebras. Then, the following induced sequence of Leibniz algebras $\mathfrak{a} \curlywedge \mathfrak{g} \to \mathfrak{g} \curlywedge \mathfrak{g} \to \mathfrak{h} \curlywedge \mathfrak{h} \to 0$ is exact.*

Proof. Since the images of the induced homomorphisms $\mathfrak{a} \curlywedge \mathfrak{g} \to \mathfrak{g} \curlywedge \mathfrak{g}$ and $\mathfrak{g} \curlywedge \mathfrak{a} \to \mathfrak{g} \curlywedge \mathfrak{g}$ are the same, the statement follows immediately from the exactness of (5.3.1). □

Proposition 5.3.10. *Let $\mathfrak{f}$ be a free Leibniz algebra over a set X. Then $\theta_{\mathfrak{f},\mathfrak{f}}$ is injective.*

Proof. We will consider the epimorphism $\theta_{\mathfrak{f},\mathfrak{f}}\colon \mathfrak{f} \curlywedge \mathfrak{f} \twoheadrightarrow [\mathfrak{f},\mathfrak{f}]$ and show that it is an isomorphism. Using the same notations as in Subsection 5.2.1, suppose $[M_X, M_X]$ denotes the subset $\{[x,y] \mid x,y \in M_X\}$ of M_X and $\mathbb{K}[M_X, M_X]$ denotes the free vector space over the set $[M_X, M_X]$. Then,

$$[\mathfrak{f},\mathfrak{f}] = \frac{\mathbb{K}[M_X, M_X]}{\mathcal{N}(S_X)}.$$

Note that for each element $m \in [M_X, M_X]$ there are unique x and y in M_X such that $m = [x,y]$. Therefore, the following map $\tau\colon \mathbb{K}[M_X, M_X] \to \mathfrak{f} \curlywedge \mathfrak{f}$, given by $[x,y] \mapsto x \curlywedge y$ for each $x, y \in M(X)$, is well defined. We have

$$[x,[y,z]] - [[x,y],z] + [[x,z],y] \overset{\tau}{\mapsto} x \curlywedge [y,z] - [x,y] \curlywedge z + [x,z] \curlywedge y = 0,$$

for each $x, y, z \in M_X$. Moreover, if $m = \sum\limits_{i=1}^{n} k_i x_i \in S_X$ with $k_1, \dots, k_n \in \mathbb{K}$ and $x_1, \dots, x_n \in M_X$, then we have:

$$[x,m] \overset{\tau}{\mapsto} \sum_{i=1}^{n} k_i (x \curlywedge x_i) = x \curlywedge \Big(\sum_{i=1}^{n} k_i x_i\Big) = 0,$$

$$[m,x] \overset{\tau}{\mapsto} \sum_{i=1}^{n} k_i (x_i \curlywedge x) = \Big(\sum_{i=1}^{n} k_i x_i\Big) \curlywedge x = 0,$$

for each $x \in M_X$. As a result we have that $\tau(\mathcal{N}(S_X)) = 0$. Thus, τ induces a well-defined linear map $\tau^*\colon [\mathfrak{f},\mathfrak{f}] \to \mathfrak{f} \curlywedge \mathfrak{f}$. Furthermore, $\tau^* \circ \theta_{\mathfrak{f},\mathfrak{f}} = 1_{\mathfrak{f} \curlywedge \mathfrak{f}}$ and $\theta_{\mathfrak{f},\mathfrak{f}} \circ \tau^* = 1_{[\mathfrak{f},\mathfrak{f}]}$. This completes the proof. □

Corollary 5.3.11. *Let $0 \to \mathfrak{r} \to \mathfrak{f} \to \mathfrak{g} \to 0$ be a free presentation of a Leibniz algebra $\mathfrak{g}$. Then there is an isomorphism*

$$\mathfrak{g} \curlywedge \mathfrak{g} \cong [\mathfrak{f},\mathfrak{f}]/[\mathfrak{r},\mathfrak{f}].$$

Proof. This follows from Lemma 5.3.9 and Proposition 5.3.10. □

Theorem 5.3.12. *Let $\mathfrak{g}$ be a Leibniz algebra. Then there is an isomorphism*

$$HL_2(\mathfrak{g}) \cong \mathsf{Ker}\Big(\theta_{\mathfrak{g},\mathfrak{g}} \colon \mathfrak{g} \curlywedge \mathfrak{g} \to \mathfrak{g}\Big).$$

Proof. Let $0 \to \mathfrak{r} \to \mathfrak{f} \to \mathfrak{g} \to 0$ be a free presentation of $\mathfrak{g}$. By the Hopf formula we have

$$HL_2(\mathfrak{g}) \cong \mathsf{Ker}\Big([\mathfrak{f},\mathfrak{f}]/[\mathfrak{r},\mathfrak{f}] \to \mathfrak{g}\Big).$$

Thus, Corollary 5.3.11 completes the proof. □

Proposition 5.3.13. *Let $0 \to \mathfrak{a} \to \mathfrak{g} \to \mathfrak{h} \to 0$ be an extension of Leibniz algebras. Then we have the following exact sequence*

$$\begin{aligned}\mathsf{Ker}\big(\theta_{\mathfrak{a},\mathfrak{g}} \colon \mathfrak{a} \curlywedge \mathfrak{g} \to \mathfrak{a}\big) \to HL_2(\mathfrak{g}) \to HL_2(\mathfrak{h})& \\ \to \mathfrak{a}/[\mathfrak{a},\mathfrak{g}] \to HL_1(\mathfrak{g}) \to HL_1(\mathfrak{h}) \to 0.&\end{aligned}$$

Proof. By Lemma 5.3.9 we have the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} & & \mathfrak{a} \curlywedge \mathfrak{g} & \longrightarrow & \mathfrak{g} \curlywedge \mathfrak{g} & \longrightarrow & \mathfrak{h} \curlywedge \mathfrak{h} & \longrightarrow & 0 \\ & & \downarrow {\scriptstyle \theta_{\mathfrak{a},\mathfrak{g}}} & & \downarrow {\scriptstyle \theta_{\mathfrak{g},\mathfrak{g}}} & & \downarrow {\scriptstyle \theta_{\mathfrak{h},\mathfrak{h}}} & & \\ 0 & \longrightarrow & \mathfrak{a} & \longrightarrow & \mathfrak{g} & \longrightarrow & \mathfrak{h} & \longrightarrow & 0. \end{array}$$

Now the Snake Lemma and Theorem 5.3.12 yield the exact sequence. □

Let $\mathfrak{g} \bullet \mathfrak{g}$ denote the vector space $\mathrm{Coker}(\mathfrak{g}\otimes\mathfrak{g} \otimes \mathfrak{g} \xrightarrow{d} \mathfrak{g} \otimes \mathfrak{g})$, where d is the boundary map in $CL_*(\mathfrak{g})$. Let $\delta \colon \mathfrak{g} \bullet \mathfrak{g} \to \mathfrak{g} \curlywedge \mathfrak{g}$ be a linear map given by $x \bullet y \mapsto x \curlywedge y$, where $x \bullet y$ denotes the coset of $x \otimes y \in \mathfrak{g} \otimes \mathfrak{g}$ into $\mathfrak{g} \bullet \mathfrak{g}$. It is easy to check that δ is well defined.

Proposition 5.3.14. *The linear map $\delta \colon \mathfrak{g} \bullet \mathfrak{g} \to \mathfrak{g} \curlywedge \mathfrak{g}$ is an isomorphism of vector spaces.*

Proof. We have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathsf{Ker}\, d' & \longrightarrow & \mathfrak{g} \bullet \mathfrak{g} & \xrightarrow[d']{} & \mathfrak{g} & \longrightarrow & \mathfrak{g}/[\mathfrak{g},\mathfrak{g}] \\
 & & \downarrow & & \downarrow{\scriptstyle \delta} & & \| & & \| \\
0 & \longrightarrow & \mathsf{Ker}\, \theta_{\mathfrak{g},\mathfrak{g}} & \longrightarrow & \mathfrak{g} \curlywedge \mathfrak{g} & \xrightarrow[\theta_{\mathfrak{g},\mathfrak{g}}]{} & \mathfrak{g} & \longrightarrow & \mathfrak{g}/[\mathfrak{g},\mathfrak{g}],
\end{array}$$

where d' is given by $x \bullet y \mapsto [x, y]$ for each $x, y \in \mathfrak{g}$. Since $HL_2(\mathfrak{g}) = \mathsf{Ker}\, d'$, by Theorem 5.3.12 we have an isomorphism $\mathsf{Ker}\, d' \cong \mathsf{Ker}\, \theta_{\mathfrak{g},\mathfrak{g}}$. It is easy to verify that this isomorphism is induced by δ. Hence, the above diagram proves the proposition. □

Remark 5.3.15. It is shown in [21] that the vector space $\mathfrak{g} \bullet \mathfrak{g}$ has the Leibniz algebra structure given by

$$[x \bullet y, x' \bullet y'] = [x, y] \bullet [x', y'],$$

for each $x, y, x', y' \in \mathfrak{g}$. This fact results from the previous proposition, because in $\mathfrak{g} \curlywedge \mathfrak{g}$ we have that $[x \curlywedge y, x' \curlywedge y'] = [x, y] \curlywedge [x', y']$.

5.4 Third homology and the eight term exact sequence

In this section we will use the method developed in [12] to construct an eight term exact sequence in Leibniz homology.

Lemma 5.4.1. *Let* $0 \to \mathfrak{a} \to \mathfrak{g} \xrightarrow{\tau} \mathfrak{h} \to 0$ *be a split extension of Leibniz algebras, i.e. there is a homomorphism of Leibniz algebras* $\sigma\colon \mathfrak{h} \to \mathfrak{g}$ *such that* $\tau \circ \sigma = 1_{\mathfrak{h}}$. *Then the induced homomorphism of Leibniz algebras* $\mathfrak{a} \curlywedge \mathfrak{g} \to \mathfrak{g} \curlywedge \mathfrak{g}$ *is injective.*

Proof. Denote the induced homomorphism of Leibniz algebras $\mathfrak{a} \curlywedge \mathfrak{g} \to \mathfrak{g} \curlywedge \mathfrak{g}$ by α. We shall show that there exists a linear map $\mathfrak{g} \curlywedge \mathfrak{g} \to \mathfrak{a} \curlywedge \mathfrak{g}$ which is a $\mathbb{K}$-linear splitting for α. For each element $x \in \mathfrak{g}$ there are unique $a \in \mathfrak{a}$ and $h \in \mathfrak{h}$ such that $x = a + \sigma(h)$. Let $\beta\colon \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{a} \curlywedge \mathfrak{g}$ be a linear map given by

$(a+\sigma(h))\otimes(a'+\sigma(h')) \overset{\beta}{\mapsto} a \curlywedge \sigma(h') + a \curlywedge a' + \sigma(h) \curlywedge a'$ for each $a, a' \in \mathfrak{a}$ and $h, h' \in \mathfrak{h}$. It is easy to check that β is a well-defined $\mathbb{K}$-linear map and that $\beta(\operatorname{Im} d) = 0$, where $d\colon \mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g} \to \mathfrak{g}\otimes\mathfrak{g}$ is the boundary map in $CL_*(\mathfrak{g})$. Thus, β induces the linear map $\bar{\beta}\colon \mathfrak{g}\bullet\mathfrak{g} \to \mathfrak{a}\curlywedge\mathfrak{g}$. Now, let $\delta\colon \mathfrak{g}\bullet\mathfrak{g} \to \mathfrak{g}\curlywedge\mathfrak{g}$ be the linear map defined as in Section 5.3. Then, the linear map $\bar{\beta}\delta^{-1}\colon \mathfrak{g}\curlywedge\mathfrak{g} \to \mathfrak{a}\curlywedge\mathfrak{g}$ is such that $\bar{\beta}\delta^{-1}\alpha = 1_{\mathfrak{a}\curlywedge\mathfrak{g}}$. Thus, α is injective. □

Theorem 5.4.2. *Let $0 \to \mathfrak{r} \to \mathfrak{f} \to \mathfrak{g} \to 0$ be a free presentation of a Leibniz algebra $\mathfrak{g}$. Then there is an isomorphism*

$$HL_3(\mathfrak{g}) \cong \operatorname{Ker}\Big(\theta_{\mathfrak{r},\mathfrak{f}}\colon \mathfrak{r}\curlywedge\mathfrak{f} \to \mathfrak{r}\Big).$$

Proof. According to Remark 5.2.4, for computing $HL_*(\mathfrak{g})$ we can use an exact augmented simplicial Leibniz algebra

$$\cdots\ \mathfrak{f}_2 \overset{d_0^2}{\underset{d_2^2}{\rightrightarrows\!\!\to}} \mathfrak{f}_1 \ \overset{d_0^1}{\underset{d_1^1}{\rightrightarrows}} \mathfrak{f}_0 \overset{d_0^0}{\to} \mathfrak{g}$$

such that $\mathfrak{f}_i$ is a free Leibniz algebra over a set, for each $i \geqslant 0$, $\mathfrak{f}_0 = \mathfrak{f}$ and $\operatorname{Ker} d_0^0 = \mathfrak{r}$. Then, the long exact homotopy sequence derived from the following short exact sequence of simplicial Leibniz algebras

$$0 \to [\mathfrak{f}_*, \mathfrak{f}_*] \to \mathfrak{f}_* \to \mathfrak{Ab}(\mathfrak{f}_*) \to 0,$$

implies that $HL_3(\mathfrak{g})$ is isomorphic to the first homotopy group of the following simplicial Leibniz algebra

$$\cdots\ [\mathfrak{f}_2, \mathfrak{f}_2] \overset{d_0^2}{\underset{d_2^2}{\rightrightarrows\!\!\to}} [\mathfrak{f}_1, \mathfrak{f}_1] \ \overset{d_0^1}{\underset{d_1^1}{\rightrightarrows}} [\mathfrak{f}_0, \mathfrak{f}_0].$$

Hence,

$$HL_3(\mathfrak{g}) \cong \operatorname{Ker} d_0^1 \cap \operatorname{Ker} d_1^1 \cap [\mathfrak{f}_1, \mathfrak{f}_1]/d_2^2\big(\operatorname{Ker} d_0^2 \cap \operatorname{Ker} d_1^2 \cap [\mathfrak{f}_2, \mathfrak{f}_2]\big).$$

Since $HL_2(\mathfrak{f}_0) = 0$ and $HL_3(\mathfrak{f}_1) = 0$, using Hopf formulas we have

$$\operatorname{Ker} d_0^1 \cap [\mathfrak{f}_1, \mathfrak{f}_1] = [\operatorname{Ker} d_0^1, \mathfrak{f}_1],$$
$$\operatorname{Ker} d_0^2 \cap \operatorname{Ker} d_1^2 \cap [\mathfrak{f}_2, \mathfrak{f}_2] = [\operatorname{Ker} d_0^2 \cap \operatorname{Ker} d_1^2, \mathfrak{f}_2] + [\operatorname{Ker} d_0^2, \operatorname{Ker} d_1^2].$$

Therefore,

$$\begin{aligned} HL_3(\mathfrak{g}) &\cong \mathsf{Ker}\, d_1^1 \cap [\mathsf{Ker}\, d_0^1, \mathfrak{f}_1]/d_2^2\big([\mathsf{Ker}\, d_0^2 \cap \mathsf{Ker}\, d_1^2, \mathfrak{f}_2] + [\mathsf{Ker}\, d_0^2, \mathsf{Ker}\, d_1^2]\big) \\ &= \mathsf{Ker}\, d_1^1 \cap [\mathsf{Ker}\, d_0^1, \mathfrak{f}_1]/\big([d_2^2(\mathsf{Ker}\, d_0^2 \cap \mathsf{Ker}\, d_1^2), d_2^2(\mathfrak{f}_2)] \\ &\quad + [d_2^2(\mathsf{Ker}\, d_0^2), d_2^2(\mathsf{Ker}\, d_1^2)]\big) \\ &= \mathsf{Ker}\, d_1^1 \cap [\mathsf{Ker}\, d_0^1, \mathfrak{f}_1]/\big([\mathsf{Ker}\, d_0^1 \cap \mathsf{Ker}\, d_1^1, \mathfrak{f}_1] + [\mathsf{Ker}\, d_0^1, \mathsf{Ker}\, d_1^1]\big). \end{aligned}$$

Since $d_1^1\big([\mathsf{Ker}\, d_0^1 \cap \mathsf{Ker}\, d_1^1, \mathfrak{f}_1] + [\mathsf{Ker}\, d_0^1, \mathsf{Ker}\, d_1^1]\big) = 0$, we get

$$HL_3(\mathfrak{g}) \cong \mathsf{Ker}\left(\frac{[\mathsf{Ker}\, d_0^1, \mathfrak{f}_1]}{[\mathsf{Ker}\, d_0^1 \cap \mathsf{Ker}\, d_1^1, \mathfrak{f}_1] + [\mathsf{Ker}\, d_0^1, \mathsf{Ker}\, d_1^1]} \xrightarrow{d_1^1} [\mathfrak{f}_0, \mathfrak{f}_0]\right). \tag{5.4.1}$$

Furthermore, since $0 \to \mathsf{Ker}\, d_1^1 \to \mathfrak{f}_1 \xrightarrow{d_1^1} \mathfrak{f}_0 \to 0$ is a free presentation of $\mathfrak{f}_0$ which splits, by Proposition 5.3.10 and Lemma 5.4.1 the following map $\mathsf{Ker}\, d_1^1 \curlywedge \mathfrak{f}_1 \to [\mathsf{Ker}\, d_1^1, \mathfrak{f}_1]$, defined by $x \curlywedge y \mapsto [x, y]$, $y \curlywedge x \mapsto [y, x]$, for all $x \in \mathsf{Ker}\, d_1^1$ and $y \in \mathfrak{f}_1$, is an isomorphism. Therefore,

$$\frac{[\mathsf{Ker}\, d_0^1, \mathfrak{f}_1]}{[\mathsf{Ker}\, d_0^1 \cap \mathsf{Ker}\, d_1^1, \mathfrak{f}_1] + [\mathsf{Ker}\, d_0^1, \mathsf{Ker}\, d_1^1]} \cong \frac{\mathsf{Ker}\, d_0^1 \curlywedge \mathfrak{f}_1}{\big((\mathsf{Ker}\, d_0^1 \cap \mathsf{Ker}\, d_1^1) \curlywedge \mathfrak{f}_1\big) + \big(\mathsf{Ker}\, d_0^1 \curlywedge \mathsf{Ker}\, d_1^1\big)}.$$

Hence, the exact sequences $0 \to \mathsf{Ker}\, d_0^1 \cap \mathsf{Ker}\, d_1^1 \to \mathsf{Ker}\, d_0^1 \to \mathsf{Ker}\, d_0^0 \to 0$ and $0 \to \mathsf{Ker}\, d_1^1 \to \mathfrak{f}_1 \to \mathfrak{f}_0 \to 0$, and (5.3.1) imply that

$$\frac{[\mathsf{Ker}\, d_0^1, \mathfrak{f}_1]}{[\mathsf{Ker}\, d_0^1 \cap \mathsf{Ker}\, d_1^1, \mathfrak{f}_1] + [\mathsf{Ker}\, d_0^1, \mathsf{Ker}\, d_1^1]} \cong \mathsf{Ker}\, d_0^0 \curlywedge \mathfrak{f}_0 = \mathfrak{r} \curlywedge \mathfrak{f}. \tag{5.4.2}$$

Now (5.4.1) and (5.4.2) complete the proof. □

Proposition 5.4.3. *Let* $0 \to \mathfrak{a} \to \mathfrak{g} \xrightarrow{\tau} \mathfrak{h} \to 0$ *be an extension of Leibniz algebras. Then we have the following exact sequence*

$$\begin{aligned} HL_3(\mathfrak{g}) \to HL_3(\mathfrak{h}) &\to \mathsf{Ker}\big(\theta_{\mathfrak{a},\mathfrak{g}} \colon \mathfrak{a} \curlywedge \mathfrak{g} \to \mathfrak{a}\big) \to HL_2(\mathfrak{g}) \to HL_2(\mathfrak{h}) \\ &\to \mathfrak{a}/[\mathfrak{a}, \mathfrak{g}] \to HL_1(\mathfrak{g}) \to HL_1(\mathfrak{h}) \to 0. \end{aligned}$$

Proof. Any free presentation $0 \to \mathfrak{r} \to \mathfrak{f} \xrightarrow{\rho} \mathfrak{g} \to 0$ of $\mathfrak{g}$ produces a free presentation $0 \to \mathfrak{s} \to \mathfrak{f} \xrightarrow{\tau\circ\rho} \mathfrak{h} \to 0$ of $\mathfrak{h}$ and an extension $0 \to \mathfrak{r} \to \mathfrak{s} \to \mathfrak{a} \to 0$ of Leibniz algebras. By (5.3.1) we have the following exact sequence

$$\mathfrak{s} \curlywedge \mathfrak{r} \times \mathfrak{r} \curlywedge \mathfrak{f} \to \mathfrak{s} \curlywedge \mathfrak{f} \to \mathfrak{a} \curlywedge \mathfrak{g} \to 0.$$

This sequence yields the following exact sequence

$$\mathfrak{r} \curlywedge \mathfrak{f} \to \mathfrak{s} \curlywedge \mathfrak{f} \to \mathfrak{a} \curlywedge \mathfrak{g} \to 0.$$

Thus, we have the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} & & \mathfrak{r} \curlywedge \mathfrak{f} & \longrightarrow & \mathfrak{s} \curlywedge \mathfrak{f} & \longrightarrow & \mathfrak{a} \curlywedge \mathfrak{g} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \theta_{\mathfrak{r},\mathfrak{f}}} & & \downarrow{\scriptstyle \theta_{\mathfrak{s},\mathfrak{f}}} & & \downarrow{\scriptstyle \theta_{\mathfrak{a},\mathfrak{g}}} & & \\ 0 & \longrightarrow & \mathfrak{r} & \longrightarrow & \mathfrak{s} & \longrightarrow & \mathfrak{a} & \longrightarrow & 0. \end{array}$$

The Snake Lemma and Theorem 5.4.2 imply the following exact sequence

$$HL_3(\mathfrak{g}) \to HL_3(\mathfrak{h}) \to \mathsf{Ker}\left(\theta_{\mathfrak{a},\mathfrak{g}} \colon \mathfrak{a} \curlywedge \mathfrak{g} \to \mathfrak{a}\right) \to \mathfrak{r}/[\mathfrak{r},\mathfrak{f}] \to \mathfrak{s}/[\mathfrak{s},\mathfrak{f}].$$

It is easy to see that

$$\mathsf{Im}\left(\mathsf{Ker}\left(\theta_{\mathfrak{a},\mathfrak{g}} \colon \mathfrak{a} \curlywedge \mathfrak{g} \to \mathfrak{a}\right) \to \mathfrak{r}/[\mathfrak{r},\mathfrak{f}]\right) \subseteq \frac{\mathfrak{r} \cap [\mathfrak{f},\mathfrak{f}]}{[\mathfrak{r},\mathfrak{f}]}.$$

Therefore, we have an exact sequence:

$$HL_3(\mathfrak{g}) \to HL_3(\mathfrak{h}) \to \mathsf{Ker}\left(\theta_{\mathfrak{a},\mathfrak{g}} \colon \mathfrak{a} \curlywedge \mathfrak{g} \to \mathfrak{a}\right) \to \frac{\mathfrak{r} \cap [\mathfrak{f},\mathfrak{f}]}{[\mathfrak{r},\mathfrak{f}]} \to \frac{\mathfrak{s} \cap [\mathfrak{f},\mathfrak{f}]}{[\mathfrak{s},\mathfrak{f}]}.$$

Using the Hopf formula we get an exact sequence:

$$HL_3(\mathfrak{g}) \to HL_3(\mathfrak{h}) \to \mathsf{Ker}\left(\theta_{\mathfrak{a},\mathfrak{g}} \colon \mathfrak{a} \curlywedge \mathfrak{g} \to \mathfrak{a}\right) \to HL_2(\mathfrak{g}) \to HL_2(\mathfrak{h}).$$

The rest of the proof follows from Proposition 5.3.13. □

Corollary 5.4.4. *(see [9]) Let* $0 \to \mathfrak{a} \to \mathfrak{g} \to \mathfrak{h} \to 0$ *be a central extension of Leibniz algebras, i.e.* $[a,x] = [x,a] = 0$ *for all* $a \in \mathfrak{a}$ *and* $x \in \mathfrak{g}$*. Then there is the following exact sequence*

$$\begin{aligned} HL_3(\mathfrak{g}) \to HL_3(\mathfrak{h}) \to \mathrm{Coker}\left(\mathfrak{a} \otimes \mathfrak{a} \xrightarrow{\eta} \mathfrak{a} \otimes \frac{\mathfrak{g}}{[\mathfrak{g},\mathfrak{g}]} \oplus \frac{\mathfrak{g}}{[\mathfrak{g},\mathfrak{g}]} \otimes \mathfrak{a}\right) \\ \to HL_2(\mathfrak{g}) \to HL_2(\mathfrak{h}) \to \mathfrak{a} \to HL_1(\mathfrak{g}) \to HL_1(\mathfrak{h}) \to 0, \end{aligned}$$

where $\eta \colon \mathfrak{a} \otimes \mathfrak{a} \to \mathfrak{a} \otimes \frac{\mathfrak{g}}{[\mathfrak{g},\mathfrak{g}]} \oplus \frac{\mathfrak{g}}{[\mathfrak{g},\mathfrak{g}]} \otimes \mathfrak{a}$ *is given by* $a \otimes b \mapsto (a \otimes \overline{b}, -\overline{a} \otimes b)$*, where* $\overline{a} = a + [\mathfrak{g},\mathfrak{g}]$ *and* $\overline{b} = b + [\mathfrak{g},\mathfrak{g}]$ *for each* $a, b \in \mathfrak{a}$*.*

Proof. Under the required conditions, the Leibniz algebras $\mathfrak{a}$ and $\mathfrak{g}$ act trivially on each other. Then, by [18, Proposition 4.2], we have a natural isomorphism

$$\mathfrak{a} \star \mathfrak{g} \cong \mathfrak{a} \otimes \frac{\mathfrak{g}}{[\mathfrak{g}, \mathfrak{g}]} \oplus \frac{\mathfrak{g}}{[\mathfrak{g}, \mathfrak{g}]} \otimes \mathfrak{a}.$$

Since $\mathfrak{a} \curlywedge \mathfrak{g}$ is obtained from $\mathfrak{a} \star \mathfrak{g}$ by killing the elements of the form $a * i(b) - i(a) * b$, where $a, b \in \mathfrak{a}$ and $i \colon \mathfrak{a} \to \mathfrak{g}$ is the natural inclusion, we get an isomorphism

$$\mathfrak{a} \curlywedge \mathfrak{g} \cong \operatorname{Coker}\Big(\eta \colon \mathfrak{a} \otimes \mathfrak{a} \to \mathfrak{a} \otimes \frac{\mathfrak{g}}{[\mathfrak{g}, \mathfrak{g}]} \oplus \frac{\mathfrak{g}}{[\mathfrak{g}, \mathfrak{g}]} \otimes \mathfrak{a}\Big).$$

Then, by Proposition 5.4.3 we conclude the required result. □

5.5 Relationship to the universal quadratic functor

In this section $\mathbb{K}$ is a commutative ring with identity (not necessarily a field). Keeping in mind Remark 5.3.6, we will use only those constructions and facts from the previous sections, which do not require $\mathbb{K}$ to be a field.

In the case of Lie algebras, there is a connection between the non-abelian exterior product of Lie algebras and Whitehead's universal quadratic functor ([15]). We can observe it in the Leibniz algebras case too.

Definition 5.5.1 ([29])**.** Let A be a $\mathbb{K}$-module and consider the endofunctor that sends A to the $\mathbb{K}$-module generated by the symbols $\gamma(a)$ with $a \in A$, quotient by the submodule generated by

$$\begin{gathered} k^2\gamma(a) = \gamma(ka), \\ \gamma(a+b+c) + \gamma(a) + \gamma(b) + \gamma(c) = \gamma(a+b) + \gamma(a+c) + \gamma(b+c), \\ \gamma(ka+b) + k\gamma(a) + k\gamma(b) = k\gamma(a+b) + \gamma(ka) + \gamma(b), \end{gathered}$$

for all $k \in \mathbb{K}$ and $a, b, c \in A$. This functor denoted by Γ is called *universal quadratic functor.*

Proposition 5.5.2 ([29])**.** *Let I be a well-ordered set and A be a free $\mathbb{K}$-module with basis $\{e_i\}_{i \in I}$. Then $\Gamma(A)$ is a free $\mathbb{K}$-module with basis*

$$\{\gamma(e_i)\}_{i \in I} \cup \{\gamma(e_i + e_j) - \gamma(e_i) - \gamma(e_j)\}_{i<j}$$

Let $\eta\colon \mathfrak{m} \to \mathfrak{g}$ and $\mu\colon \mathfrak{n} \to \mathfrak{g}$ be two crossed modules of Leibniz algebras. As we know there are induced actions of $\mathfrak{m}$ and $\mathfrak{n}$ on each other via the action of $\mathfrak{g}$. Let $\mathfrak{m} \times_{\mathfrak{g}} \mathfrak{n} = \{(m,n) \mid \eta(m) = \mu(n)\}$ be the pullback of η and μ. It is a Leibniz subalgebra of $\mathfrak{m} \oplus \mathfrak{n}$. Let $\langle \mathfrak{m}, \mathfrak{n} \rangle = \{(\tau_{\mathfrak{m}}(x), \tau_{\mathfrak{n}}(x)) \mid x \in \mathfrak{m} \star \mathfrak{n}\}$, where $\tau_{\mathfrak{m}}\colon \mathfrak{m} \star \mathfrak{n} \to \mathfrak{m}$ and $\tau_{\mathfrak{n}}\colon \mathfrak{m} \star \mathfrak{n} \to \mathfrak{n}$ are homomorphisms introduced in Subsection 5.3.2.

Proposition 5.5.3. *$\langle \mathfrak{m}, \mathfrak{n} \rangle$ is an ideal of $\mathfrak{m} \times_{\mathfrak{g}} \mathfrak{n}$ and the quotient $(\mathfrak{m} \times_{\mathfrak{g}} \mathfrak{n})/\langle \mathfrak{m}, \mathfrak{n} \rangle$ is abelian.*

Proof. The assertion that $\langle \mathfrak{m}, \mathfrak{n} \rangle$ is an ideal of $\mathfrak{m} \times_{\mathfrak{g}} \mathfrak{n}$ follows by straightforward calculations. For instance, given any $m \in \mathfrak{m}$, $n \in \mathfrak{n}$ and $(m', n') \in \mathfrak{m} \times_{\mathfrak{g}} \mathfrak{n}$ we get

$$\begin{aligned}
[(\tau_{\mathfrak{m}}(m * n), \tau_{\mathfrak{n}}(m * n)), (m', n')] &= [(m^n, {}^m n), (m', n')] \\
&= ([m^n, m'], [{}^m n, n']) \\
&= ((m^n)^{\eta(m')}, {}^{\mu({}^m n)} n') \\
&= ((m^n)^{n'}, {}^{(m^n)} n') \\
&= (\tau_{\mathfrak{m}}(m^n * n'), \tau_{\mathfrak{n}}(m^n * n')).
\end{aligned}$$

Now take any (m,n), $(m',n') \in \mathfrak{m} \times_{\mathfrak{g}} \mathfrak{n}$, then we have

$$\begin{aligned}
[(m,n),(m',n')] &= ([m,m'],[n,n']) = (m^{m'}, {}^n n') \\
&= (m^{n'}, {}^m n') = (\tau_{\mathfrak{m}}(m * n'), \tau_{\mathfrak{n}}(m * n')),
\end{aligned}$$

showing that $(\mathfrak{m} \times_{\mathfrak{g}} \mathfrak{n})/\langle \mathfrak{m}, \mathfrak{n} \rangle$ is abelian. □

In the subsequent statements, given a $\mathbb{K}$-module A, we consider the $\mathbb{K}$-module $\Gamma(A)$ as an abelian Leibniz algebra.

Proposition 5.5.4. *There is a well-defined homomorphism of Leibniz algebras*

$$\Gamma\left(\frac{\mathfrak{m} \times_{\mathfrak{g}} \mathfrak{n}}{\langle \mathfrak{m}, \mathfrak{n} \rangle}\right) \xrightarrow{\psi} \mathfrak{m} \star \mathfrak{n},$$

given by $\psi(\gamma((m,n) + \langle \mathfrak{m}, \mathfrak{n} \rangle)) = m * n - n * m$.

Proof. It is easy to check that ψ preserves the defining relations of Γ. Thus, it suffices to show that $(m' + \tau_{\mathfrak{m}}(x)) * (n' + \tau_{\mathfrak{n}}(x)) - (n' + \tau_{\mathfrak{n}}(x)) * (m' + \tau_{\mathfrak{m}}(x)) = m' * n' - n' * m'$ for each $x \in \mathfrak{m} \star \mathfrak{n}$. This reduces to prove that $m' * {}^{m}n - {}^{m}n * m' + m^{n} * n' - n' * m^{n} = 0$. Using the defining relations of the non-abelian tensor product we have

$$\begin{aligned}
m' * {}^{m}n - {}^{m}n * m' + m^{n} * n' - n' * m^{n}& \\
=&[m', m] * n - {}^{m'}n * m - m * n^{m'} - [m, m'] * n \\
&+ m^{n'} * n + m * [n, n'] - {}^{n'}m * n + [n', n] * m \\
=&[m', m] * n - [n', n] * m - m * [n, n'] - [m, m'] * n \\
&+ [m, m'] * n + m * [n, n'] + [m', m] * n - [n', n] * m = 0.
\end{aligned}$$

By Proposition 5.3.4 we know that $\operatorname{Im}\psi$ is contained in the centre of $\mathfrak{m} \star \mathfrak{n}$, so ψ is a homomorphism of Leibniz algebras. □

It is clear that $\operatorname{Im}\psi$ is contained in $\operatorname{Ker}(\pi\colon \mathfrak{m} \star \mathfrak{n} \to \mathfrak{m} \curlywedge \mathfrak{n})$, where π is the canonical projection. But the following sequence

$$\Gamma\left(\frac{\mathfrak{m} \times_{\mathfrak{g}} \mathfrak{n}}{\langle \mathfrak{m}, \mathfrak{n} \rangle}\right) \xrightarrow{\psi} \mathfrak{m} \star \mathfrak{n} \xrightarrow{\pi} \mathfrak{m} \curlywedge \mathfrak{n} \longrightarrow 0,$$

is not exact in many cases. Nevertheless, we have the following

Proposition 5.5.5. *There is an exact sequence of Leibniz algebras*

$$\Gamma\left(\frac{\mathfrak{m} \times_{\mathfrak{g}} \mathfrak{n}}{\langle \mathfrak{m}, \mathfrak{n} \rangle} \oplus \frac{\mathfrak{m} \times_{\mathfrak{g}} \mathfrak{n}}{\langle \mathfrak{m}, \mathfrak{n} \rangle}\right) \xrightarrow{\widetilde{\psi}} \mathfrak{m} \star \mathfrak{n} \xrightarrow{\pi} \mathfrak{m} \curlywedge \mathfrak{n} \longrightarrow 0,$$

where $\widetilde{\psi}(\gamma((m, n) + \langle \mathfrak{m}, \mathfrak{n} \rangle, (m', n') + \langle \mathfrak{m}, \mathfrak{n} \rangle)) = m * n' - n * m'$.

Proof. Like in Proposition 5.5.4, the crucial part of the proof is to show that $(m + \tau_{\mathfrak{m}}(x)) * (n' + \tau_{\mathfrak{n}}(x')) - (n + \tau_{\mathfrak{n}}(x)) * (m' + \tau_{\mathfrak{m}}(x')) = m * n' - n * m'$ for all $x, x' \in \mathfrak{m} \star \mathfrak{n}$. Let $x = m_1 * n_1$ and $x' = m'_1 * n'_1$, then proving that $m * {}^{m'_1}n'_1 + m_1^{n_1} * n' - n * m_1'^{n'_1} - {}^{m_1}n_1 * m' = 0$, will imply the result. Using the defining identities of the non-abelian tensor product, we get

$$\begin{aligned}
m * {}^{m'_1}n'_1 + m_1^{n_1} * n' - n * m_1'^{n'_1} - {}^{m_1}n_1 * m'& \\
=&[m, m'_1] * n'_1 - {}^{m}n'_1 * m'_1 + m_1 * [n_1, n'] + m_1^{n'} * n_1 \\
&+ [n, n'_1] * m'_1 - {}^{n}m'_1 * n'_1 - [m_1, m'] * n_1 - m_1 * n_1^{m'} = 0.
\end{aligned}$$

The proof is complete, since it is straightforward that in this case $\mathsf{Im}\,\widetilde{\psi} = \mathsf{Ker}\,\pi$. □

In the particular case of $\mathfrak{a}$ and $\mathfrak{b}$ being ideals of a Leibniz algebra $\mathfrak{g}$, Proposition 5.5.4 and Proposition 5.5.5 can be viewed as follows:

Corollary 5.5.6. *There is a well-defined homomorphism of Leibniz algebras*

$$\Gamma\left(\frac{\mathfrak{a}\cap\mathfrak{b}}{[\mathfrak{a},\mathfrak{b}]}\right) \xrightarrow{\psi} \mathfrak{a}\star\mathfrak{b},$$

given by $\psi(\gamma(c+[\mathfrak{a},\mathfrak{b}])) = i_1(c) * i_2(c) - i_2(c) * i_1(c)$, *for any* $c \in \mathfrak{a}\cap\mathfrak{b}$ *and the natural inclusions* $i_1\colon \mathfrak{a}\cap\mathfrak{b} \to \mathfrak{a}$, $i_2\colon \mathfrak{a}\cap\mathfrak{b} \to \mathfrak{b}$.

Corollary 5.5.7. *There is an exact sequence of Leibniz algebras*

$$\Gamma\left(\frac{\mathfrak{a}\cap\mathfrak{b}}{[\mathfrak{a},\mathfrak{b}]} \oplus \frac{\mathfrak{a}\cap\mathfrak{b}}{[\mathfrak{a},\mathfrak{b}]}\right) \xrightarrow{\widetilde{\psi}} \mathfrak{a}\star\mathfrak{b} \xrightarrow{\pi} \mathfrak{a}\curlywedge\mathfrak{b} \longrightarrow 0,$$

where $\widetilde{\psi}(c+[\mathfrak{a},\mathfrak{b}], c'+[\mathfrak{a},\mathfrak{b}]) = i_1(c) * i_2(c') - i_2(c) * i_1(c')$, *for all* $c, c' \in \mathfrak{a}\cap\mathfrak{b}$.

In the next proposition $\dfrac{\mathfrak{a}\cap\mathfrak{b}}{[\mathfrak{a},\mathfrak{b}]} \wedge \dfrac{\mathfrak{a}\cap\mathfrak{b}}{[\mathfrak{a},\mathfrak{b}]}$ denotes the exterior product of $\mathbb{K}$-modules.

Proposition 5.5.8. *There is an exact sequence of Leibniz algebras*

$$\frac{\mathfrak{a}\cap\mathfrak{b}}{[\mathfrak{a},\mathfrak{b}]} \wedge \frac{\mathfrak{a}\cap\mathfrak{b}}{[\mathfrak{a},\mathfrak{b}]} \xrightarrow{\phi} \frac{\mathfrak{a}\star\mathfrak{b}}{\mathsf{Im}\,\psi} \xrightarrow{\overline{\pi}} \mathfrak{a}\curlywedge\mathfrak{b} \longrightarrow 0,$$

where $\overline{\pi}$ *is the canonical projection and* $\phi((c+[\mathfrak{a},\mathfrak{b}]) \wedge (c'+[\mathfrak{a},\mathfrak{b}])) = i_1(c) * i_2(c') - i_2(c) * i_1(c') + \mathsf{Im}\,\psi$, *for all* $c, c' \in \mathfrak{a}\cap\mathfrak{b}$.

Proof. To check that ϕ is well defined, it suffices to verify the following identities:

$$i_1(c) * i_2[a,b] - i_2(c) * i_1[a,b] = 0,$$

$$i_1[a,b] * i_2(c) - i_2[a,b] * i_1(c) = 0,$$

for all $c \in \mathfrak{a}\cap\mathfrak{b}$, $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$. Using (3c), (3d) and (4b) we have:

$$\begin{aligned}
&i_1(c) * i_2[a,b] - i_2(c) * i_1[a,b] \\
&\quad = i_2[c,a] * i_1(b) - i_1[c,b] * i_2(a) - i_2(c) * i_1[a,b] \\
&\quad = i_1[c,b] * i_2(a) - i_2(c) * i_1[b,a] - i_1[c,b] * i_2(a) - i_2(c) * i_1[a,b] = 0.
\end{aligned}$$

Using (3d), (3b) and (4b) we have:

$$\begin{aligned}
&i_1[a,b] * i_2(c) - i_2[a,b] * i_1(c) \\
&\quad = i_2[a,c] * i_1(b) + i_2(a) * i_1[b,c] - i_2[a,b] * i_1(c) \\
&\quad = i_2(a) * i_1[c,b] + i_2(a) * i_1[b,c] = 0.
\end{aligned}$$

The proof is complete, since it is straightforward that $\mathsf{Im}\,\phi = \mathsf{Ker}\,\overline{\pi}$. □

Let $\mathfrak{g}$ be a Leibniz algebra, and let $\tau\colon \mathfrak{g} \star \mathfrak{g} \to \mathfrak{g}^{\mathrm{ab}} \otimes \mathfrak{g}^{\mathrm{ab}}$ be the homomorphism defined by $i_1(g) * i_2(g') \mapsto (g + [\mathfrak{g},\mathfrak{g}]) \otimes (g' + [\mathfrak{g},\mathfrak{g}])$, $i_2(g) * i_1(g') \mapsto 0$, for all $g, g' \in \mathfrak{g}$, where $\otimes$ denotes the tensor product of $\mathbb{K}$-modules. Then, τ induces well-defined homomorphisms $\overline{\tau}\colon \mathfrak{g} \star \mathfrak{g} \to \mathfrak{g}^{\mathrm{ab}} \curlywedge \mathfrak{g}^{\mathrm{ab}}$ and $\widetilde{\tau}\colon \dfrac{\mathfrak{g} \star \mathfrak{g}}{\mathsf{Im}\,\psi} \to \mathfrak{g}^{\mathrm{ab}} \curlywedge \mathfrak{g}^{\mathrm{ab}}$, where ψ is defined as in Proposition 5.5.4.

Proposition 5.5.9. *We have the following exact sequence of Leibniz algebras*

$$0 \longrightarrow \mathfrak{g}^{\mathrm{ab}} \curlywedge \mathfrak{g}^{\mathrm{ab}} \xrightarrow{\ \phi\ } \frac{\mathfrak{g} \star \mathfrak{g}}{\mathsf{Im}\,\psi} \xrightarrow{\ \overline{\pi}\ } \mathfrak{g} \curlywedge \mathfrak{g} \longrightarrow 0.$$

where $\overline{\pi}$ is the canonical projection and ϕ is defined as in Proposition 5.5.8. Moreover, the following map

$$\frac{\mathfrak{g} \star \mathfrak{g}}{\mathsf{Im}\,\psi} \xrightarrow{(\overline{\pi},\widetilde{\tau})} (\mathfrak{g} \curlywedge \mathfrak{g}) \oplus (\mathfrak{g}^{\mathrm{ab}} \curlywedge \mathfrak{g}^{\mathrm{ab}})$$

is an isomorphism of Leibniz algebras.

Proof. It is easy to see that $\widetilde{\tau} \circ \phi = 1_{\mathfrak{g}^{\mathrm{ab}} \curlywedge \mathfrak{g}^{\mathrm{ab}}}$. This implies both parts of the proposition. □

Proposition 5.5.10. *Let $\mathfrak{g}$ be a Leibniz algebra such that $\mathfrak{g}^{\mathrm{ab}}$ is free as a $\mathbb{K}$-module. Then there is an exact sequence of Leibniz algebras*

$$0 \longrightarrow \Gamma(\mathfrak{g}^{\mathrm{ab}}) \xrightarrow{\ \psi\ } \mathfrak{g} \star \mathfrak{g} \xrightarrow{(\pi,\overline{\tau})} (\mathfrak{g} \curlywedge \mathfrak{g}) \oplus (\mathfrak{g}^{\mathrm{ab}} \curlywedge \mathfrak{g}^{\mathrm{ab}}) \longrightarrow 0.$$

Proof. By the previous proposition it suffices to show that ψ is injective. By Proposition 5.5.2 one sees easily that the composition $\tau\circ\psi\colon \Gamma(\mathfrak{g}^{\mathrm{ab}}) \to \mathfrak{g}^{\mathrm{ab}}\otimes\mathfrak{g}^{\mathrm{ab}}$ maps a basis of $\Gamma(\mathfrak{g}^{\mathrm{ab}})$ injectively into a set of linearly independent elements. Therefore $\tau \circ \psi$ is injective and ψ is injective. □

5.6 Comparison of the second Lie and Leibniz homologies of a Lie algebra

In this section we return to the case when $\mathbb{K}$ is a field, $\mathfrak{g}$ denotes a Lie algebra and $H_2(\mathfrak{g})$ denotes the second Chevalley-Eilenberg homology of $\mathfrak{g}$. It is known that there is an epimorphism $t_{\mathfrak{g}}\colon HL_2(\mathfrak{g}) \to H_2(\mathfrak{g})$ defined in a natural way (see e.g. [27]).

Proposition 5.6.1. *There exists a vector subspace V of $\mathsf{Ker}\{t_{\mathfrak{g}}\colon HL_2(\mathfrak{g}) \to H_2(\mathfrak{g})\}$ such that we have an epimorphism $V \to \Gamma(\mathfrak{g}^{\mathrm{ab}})$. Hence, if $\mathfrak{g}$ is not a perfect Lie algebra, then $t_{\mathfrak{g}}\colon HL_2(\mathfrak{g}) \to H_2(\mathfrak{g})$ is not an isomorphism.*

Proof. Let $\mathfrak{g} \underset{Lie}{\star} \mathfrak{g}$ (resp. $\mathfrak{g} \underset{Lie}{\curlywedge} \mathfrak{g}$) denote the non-abelian tensor (resp. exterior) square of the Lie algebra $\mathfrak{g}$ (see [15]). Then we have two epimorphisms $\mathfrak{g}\curlywedge\mathfrak{g} \to \mathfrak{g} \underset{Lie}{\star} \mathfrak{g}$ and $\mathfrak{g}\curlywedge\mathfrak{g} \to \mathfrak{g} \underset{Lie}{\curlywedge} \mathfrak{g}$ defined in a natural way. Since $t_{\mathfrak{g}}\colon HL_2(\mathfrak{g}) \to H_2(\mathfrak{g})$ can be viewed as the natural homomorphism from $\mathsf{Ker}\{\mathfrak{g}\curlywedge\mathfrak{g} \to \mathfrak{g}\}$ to $\mathsf{Ker}\{\mathfrak{g} \underset{Lie}{\curlywedge} \mathfrak{g} \to \mathfrak{g}\}$, we have that $t_{\mathfrak{g}}(\mathsf{Ker}\{\mathfrak{g}\curlywedge\mathfrak{g} \to \mathfrak{g} \underset{Lie}{\curlywedge} \mathfrak{g}\}) = 0$. Let $V = \mathsf{Ker}\{\mathfrak{g}\curlywedge\mathfrak{g} \to \mathfrak{g} \underset{Lie}{\curlywedge} \mathfrak{g}\}$. Consider the following commutative diagram with exact rows:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & V & \longrightarrow & \mathfrak{g}\curlywedge\mathfrak{g} & \longrightarrow & \mathfrak{g} \underset{Lie}{\curlywedge} \mathfrak{g} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathsf{Ker}\{\mathfrak{g} \underset{Lie}{\star} \mathfrak{g} \to \mathfrak{g} \underset{Lie}{\curlywedge} \mathfrak{g}\} & \longrightarrow & \mathfrak{g} \underset{Lie}{\star} \mathfrak{g} & \longrightarrow & \mathfrak{g} \underset{Lie}{\curlywedge} \mathfrak{g} & \longrightarrow & 0.
\end{array}
$$

From this diagram we have an epimorphism $V \to \mathsf{Ker}\{\mathfrak{g} \underset{Lie}{\star} \mathfrak{g} \to \mathfrak{g} \underset{Lie}{\curlywedge} \mathfrak{g}\}$. Moreover, by [15, Proposition 17] $\mathsf{Ker}\{\mathfrak{g} \underset{Lie}{\star} \mathfrak{g} \to \mathfrak{g} \underset{Lie}{\curlywedge} \mathfrak{g}\} = \Gamma(\mathfrak{g}^{ab})$. □

Finally, we give the following result, the proof of which can be found in [6, 8].

Proposition 5.6.2. *For a perfect Lie algebra* $\mathfrak{g}$*, we have the following exact sequence:*

$$0 \to HL_2(\mathfrak{g} \underset{Lie}{\star} \mathfrak{g}) \to HL_2(\mathfrak{g}) \to H_2(\mathfrak{g}) \to 0.$$

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Chapter 6

Actor of a crossed module of Leibniz algebras

Abstract

We extend to the category of crossed modules of Leibniz algebras the notion of biderivation via the action of a Leibniz algebra. This results into a pair of Leibniz algebras which allow us to construct an object which is the actor under certain circumstances. Additionally, we give a description of an action in the category of crossed modules of Leibniz algebras in terms of equations. Finally, we check that, under the aforementioned conditions, the kernel of the canonical map from a crossed module to its actor coincides with the center and we introduce the notions of crossed module of inner and outer biderivations.

Reference

J. M. Casas, R. Fernández-Casado, X. García-Martínez, and E. Khmaladze, *Actor of a crossed module of Leibniz algebras*, preprint `arXiv:1606.04871`, 2016.

6.1 Introduction

In the category of groups it is possible to describe an action via an object called the actor, which is given by the group of automorphisms. Its analogue in the category of Lie algebras is the Lie algebra of derivations. Groups and Lie algebras are examples of categories of interest, introduced by Orzech in [13]. For these categories (see [11] for more examples), Casas, Datuashvili and Ladra [3] gave a procedure to construct an object that, under certain circumstances, plays the role of actor. For the particular case of Leibniz algebras (resp. associative algebras) that object is the Leibniz algebra of biderivations (resp. the algebra of bimultipliers).

In [12], Norrie extended the definition of actor to the 2-dimensional case by giving a description of the corresponding object in the category of crossed modules of groups. The analogue construction for the category of crossed modules of Lie algebras is given in [7]. Regarding the category of crossed modules of Leibniz algebras, it is not a category of interest, but it is equivalent to the category of cat^1-Leibniz algebras (see for example [6]), which is itself a modified category of interest in the sense of [2]. Therefore it makes sense to study representability of actions in such category under the context of modified categories of interest, as it is done in [2] for crossed modules of associative algebras.

Bearing in mind the ease of the generalization of the actor in the category of groups and Lie algebras to crossed modules, together with the role of the Leibniz algebra of biderivations, it makes sense to assume that the analogous object in the category of crossed modules of Leibniz algebras will be the actor only under certain hypotheses. In [5] the authors gave an equivalent description of an action of a crossed module of groups in terms of equations. A similar description is done for an action of a crossed module of Lie algebras (see [4]). In order to extend the notion of actor to crossed modules of Leibniz algebras, we generalize the concept of biderivation to the 2-dimensional case, describe an action in that category in terms of equations and give sufficient conditions for the described object to be the actor.

The article is organized as follows: In Section 6.2 we recall some basic definitions on actions and crossed modules of Leibniz algebras. In Section 6.3 we construct an object that extends the Leibniz algebra of biderivations to the category of crossed modules of Leibniz algebras (Theorem 6.3.9) and give a description of an action in such category in terms of equations. In Section 6.4 we find sufficient conditions for the previous object to be the actor of a given crossed module of Leibniz algebras (Theorem 6.4.3). Finally, in Section 6.5 we prove that the kernel of the canonical homomorphism from a crossed module of Leibniz algebras to its actor coincides with the center of the given crossed module. Additionally, we introduce the notions of crossed module of inner and outer biderivations and show that, given a short exact sequence in the category of crossed modules of Leibniz algebras, it can be extended to a commutative diagram including the actor and the inner and outer biderivations.

6.2 Preliminaries

In this section we recall some needed basic definitions. Throughout the paper we fix a commutative ring with unit $\mathbf{k}$. All algebras are considerer over $\mathbf{k}$.

Definition 6.2.1 ([9]). A *Leibniz algebra* $\mathfrak{p}$ is a $\mathbf{k}$-module together with a bilinear operation $[\ ,\ ]\colon \mathfrak{p} \times \mathfrak{p} \to \mathfrak{p}$, called the Leibniz bracket, which satisfies the Leibniz identity:

$$[[p_1, p_2], p_3] = [p_1, [p_2, p_3]] + [[p_1, p_3], p_2],$$

for all $p_1, p_2, p_3 \in \mathfrak{p}$.

A homomorphism of Leibniz algebras is a $\mathbf{k}$-linear map that preserves the bracket.

We denote by $\operatorname{Ann}(\mathfrak{p})$ (resp. $[\mathfrak{p}, \mathfrak{p}]$) the *annihilator* (resp. commutator) of $\mathfrak{p}$, that is the subspace of $\mathfrak{p}$ generated by

$$\{p_1 \in \mathfrak{p} \mid [p_1, p_2] = [p_2, p_1] = 0, \text{ for all } p_2 \in \mathfrak{p}\}$$

$$(\text{resp. } \{[p_1, p_2] \mid \text{for all } p_1, p_2 \in \mathfrak{p}\})$$

It is obvious that both $\operatorname{Ann}(\mathfrak{p})$ and $[\mathfrak{p}, \mathfrak{p}]$ are ideals of $\mathfrak{p}$.

Definition 6.2.2 ([10]). Let $\mathfrak{p}$ and $\mathfrak{m}$ be two Leibniz algebras. An *action* of $\mathfrak{p}$ on $\mathfrak{m}$ consists of a pair of bilinear maps, $\mathfrak{p} \times \mathfrak{m} \to \mathfrak{m}$, $(p, m) \mapsto [p, m]$ and $\mathfrak{m} \times \mathfrak{p} \to \mathfrak{m}$, $(m, p) \mapsto [m, p]$, such that

$$\begin{aligned}
[p, [m, m']] &= [[p, m], m'] - [[p, m'], m],\\
[m, [p, m']] &= [[m, p], m'] - [[m, m'], p],\\
[m, [m', p]] &= [[m, m'], p] - [[m, p], m'],\\
[m, [p, p']] &= [[m, p], p'] - [[m, p'], p],\\
[p, [m, p']] &= [[p, m], p'] - [[p, p'], m],\\
[p, [p', m]] &= [[p, p'], m] - [[p, m], p'],
\end{aligned}$$

for all $m, m' \in \mathfrak{m}$ and $p, p' \in \mathfrak{p}$.

Given an action of a Leibniz algebra $\mathfrak{p}$ on $\mathfrak{m}$, we can consider the *semidirect product* Leibniz algebra $\mathfrak{m} \rtimes \mathfrak{p}$, which consists of the $\mathbf{k}$-module $\mathfrak{m} \oplus \mathfrak{p}$ together with the Leibniz bracket given by

$$[(m,p),(m',p')] = ([m,m'] + [p,m'] + [m,p'], [p,p']),$$

for all $(m,p),(m',p') \in \mathfrak{m} \oplus \mathfrak{p}$.

Definition 6.2.3 ([10])**.** A *crossed module of Leibniz algebras* (or Leibniz crossed module, for short) $(\mathfrak{m}, \mathfrak{p}, \eta)$ is a homomorphism of Leibniz algebras $\eta\colon \mathfrak{m} \to \mathfrak{p}$ together with an action of $\mathfrak{p}$ on $\mathfrak{m}$ such that

$$\eta([p,m]) = [p,\eta(m)] \quad \text{and} \quad \eta([m,p]) = [\eta(m),p], \tag{XLb1}$$

$$[\eta(m),m'] = [m,m'] = [m,\eta(m')], \tag{XLb2}$$

for all $m,m' \in \mathfrak{m}$, $p \in \mathfrak{p}$.

A *homomorphism of Leibniz crossed modules* (φ,ψ) from $(\mathfrak{m},\mathfrak{p},\eta)$ to $(\mathfrak{n},\mathfrak{q},\mu)$ is a pair of Leibniz homomorphisms, $\varphi\colon \mathfrak{m} \to \mathfrak{n}$ and $\psi\colon \mathfrak{p} \to \mathfrak{q}$, such that they commute with η and μ and they respect the actions, that is $\varphi([p,m]) = [\psi(p),\varphi(m)]$ and $\varphi([m,p]) = [\varphi(m),\psi(p)]$ for all $m \in \mathfrak{m}$, $p \in \mathfrak{p}$.

Identity (XLb1) will be called *equivariance* and (XLb2) *Peiffer identity.* We will denote by **XLb** the category of Leibniz crossed modules and homomorphisms of Leibniz crossed modules.

Since our aim is to construct a 2-dimensional generalization of the actor in the category of Leibniz algebras, let us first recall the following definitions.

Definition 6.2.4 ([9])**.** Let $\mathfrak{m}$ be a Leibniz algebra. A *biderivation* of $\mathfrak{m}$ is a pair (d,D) of $\mathbf{k}$-linear maps $d,D\colon \mathfrak{m} \to \mathfrak{m}$ such that

$$d([m,m']) = [d(m),m'] + [m,d(m')], \tag{6.2.1}$$

$$D([m,m']) = [D(m),m'] - [D(m'),m], \tag{6.2.2}$$

$$[m,d(m')] = [m,D(m')], \tag{6.2.3}$$

for all $m,m' \in \mathfrak{m}$.

We will denote by $\operatorname{Bider}(\mathfrak{m})$ the set of all biderivations of $\mathfrak{m}$. It is a Leibniz algebra with the obvious $\mathbf{k}$-module structure and the Leibniz bracket given by

$$[(d_1,D_1),(d_2,D_2)] = (d_1d_2 - d_2d_1, D_1d_2 - d_2D_1).$$

It is not difficult to check that, given an element $m \in \mathfrak{m}$, the pair $(\mathrm{ad}(m), \mathrm{Ad}(m))$, with $\mathrm{ad}(m)(m') = -[m', m]$ and $\mathrm{Ad}(m)(m') = [m, m']$ for all $m' \in \mathfrak{m}$, is a biderivation. The pair $(\mathrm{ad}(m), \mathrm{Ad}(m))$ is called *inner biderivation* of m.

6.3 The main construction

In this section we extend to crossed modules the Leibniz algebra of biderivations. First we need to translate the notion of a biderivation of a Leibniz algebra into a biderivation between two Leibniz algebras via the action.

Definition 6.3.1. Given an action of Leibniz algebras of $\mathfrak{q}$ on $\mathfrak{n}$, the set of *biderivations* from $\mathfrak{q}$ to $\mathfrak{n}$, denoted by $\mathrm{Bider}(\mathfrak{q}, \mathfrak{n})$, consists of all the pairs (d, D) of $\mathbf{k}$-linear maps, $d, D \colon \mathfrak{q} \to \mathfrak{n}$, such that

$$d([q, q']) = [d(q), q'] + [q, d(q')], \tag{6.3.1}$$
$$D([q, q']) = [D(q), q'] - [D(q'), q], \tag{6.3.2}$$
$$[q, d(q')] = [q, D(q')], \tag{6.3.3}$$

for all $q, q' \in \mathfrak{q}$.

Given $n \in \mathfrak{n}$, the pair of $\mathbf{k}$-linear maps $(\mathrm{ad}(n), \mathrm{Ad}(n))$, where $\mathrm{ad}(n)(q) = -[q, n]$ and $\mathrm{Ad}(n)(q) = [n, q]$ for all $q \in \mathfrak{q}$, is clearly a biderivation from $\mathfrak{q}$ to $\mathfrak{n}$. Observe that $\mathrm{Bider}(\mathfrak{q}, \mathfrak{q})$, with the action of $\mathfrak{q}$ on itself defined by its Leibniz bracket, is exactly $\mathrm{Bider}(\mathfrak{q})$.

Let us assume for the rest of the article that $(\mathfrak{n}, \mathfrak{q}, \mu)$ is a Leibniz crossed module. One can easily check the following result.

Lemma 6.3.2. *Let* $(d, D) \in \mathrm{Bider}(\mathfrak{q}, \mathfrak{n})$*. Then* $(d\mu, D\mu) \in \mathrm{Bider}(\mathfrak{n})$ *and* $(\mu d, \mu D) \in \mathrm{Bider}(\mathfrak{q})$*.*

We also have the following result.

Lemma 6.3.3. *Let* (d_1, D_1)*,* $(d_2, D_2) \in \mathrm{Bider}(\mathfrak{q}, \mathfrak{n})$*. Then*

$$[D_1\mu d_2(q), q'] = [D_1\mu D_2(q), q'],$$
$$[q, D_1\mu d_2(q')] = [q, D_1\mu D_2(q')],$$

for all $q, q' \in \mathfrak{q}$*.*

Proof. Let $q, q' \in \mathfrak{q}$ and $(d_1, D_1), (d_2, D_2) \in \mathrm{Bider}(\mathfrak{q}, \mathfrak{n})$. According to the identity (6.3.3) for (d_2, D_2), $[q', d_2(q)] = [q', D_2(q)]$, so $D_1\mu([q', d_2(q)]) = D_1\mu([q', D_2(q)])$. Due to (6.3.2) and the equivariance of $(\mathfrak{q}, \mathfrak{n}, \mu)$, one can easily derive that

$$[D_1(q'), \mu d_2(q)] - [D_1\mu d_2(q), q'] = [D_1(q'), \mu D_2(q)] - [D_1\mu D_2(q), q'].$$

By the Peiffer identity and (6.3.3) for (d_2, D_2), $[D_1(q'), \mu d_2(q)] = [D_1(q'), \mu D_2(q)]$. Therefore $[D_1\mu d_2(q), q'] = [D_1\mu D_2(q), q']$.

The other identity can be proved similarly by using (6.3.1) and (6.3.3). □

$\mathrm{Bider}(\mathfrak{q}, \mathfrak{n})$ has an obvious $\mathbf{k}$-module structure. Regarding its Leibniz structure, it is described in the next proposition.

Proposition 6.3.4. $\mathrm{Bider}(\mathfrak{q}, \mathfrak{n})$ *is a Leibniz algebra with the bracket given by*

$$[(d_1, D_1), (d_2, D_2)] = (d_1\mu d_2 - d_2\mu d_1, D_1\mu d_2 - d_2\mu D_1) \tag{6.3.4}$$

for all $(d_1, D_1), (d_2, D_2) \in \mathrm{Bider}(\mathfrak{q}, \mathfrak{n})$.

Proof. It follows directly from Lemma 6.3.3. □

Now we state the following definition.

Definition 6.3.5. The set of *biderivations of the Leibniz crossed module* $(\mathfrak{n}, \mathfrak{q}, \mu)$, denoted by $\mathrm{Bider}(\mathfrak{n}, \mathfrak{q}, \mu)$, consists of all quadruples $((\sigma_1, \theta_1), (\sigma_2, \theta_2))$ such that

$$(\sigma_1, \theta_1) \in \mathrm{Bider}(\mathfrak{n}) \quad \text{and} \quad (\sigma_2, \theta_2) \in \mathrm{Bider}(\mathfrak{q}), \tag{6.3.5}$$

$$\mu\sigma_1 = \sigma_2\mu \quad \text{and} \quad \mu\theta_1 = \theta_2\mu, \tag{6.3.6}$$

$$\sigma_1([q, n]) = [\sigma_2(q), n] + [q, \sigma_1(n)], \tag{6.3.7}$$

$$\sigma_1([n, q]) = [\sigma_1(n), q] + [n, \sigma_2(q)], \tag{6.3.8}$$

$$\theta_1([q, n]) = [\theta_2(q), n] - [\theta_1(n), q], \tag{6.3.9}$$

$$\theta_1([n, q]) = [\theta_1(n), q] - [\theta_2(q), n], \tag{6.3.10}$$

$$[q, \sigma_1(n)] = [q, \theta_1(n)], \tag{6.3.11}$$

$$[n, \sigma_2(q)] = [n, \theta_2(q)], \tag{6.3.12}$$

for all $n \in \mathfrak{n}$, $q \in \mathfrak{q}$.

Given $q \in \mathfrak{q}$, it can be readily checked that $((\sigma_1^q, \theta_1^q), (\sigma_2^q, \theta_2^q))$, where

$$\sigma_1^q(n) = -[n, q], \qquad \theta_1^q(n) = [q, n],$$
$$\sigma_2^q(q') = -[q', q], \qquad \theta_2^q(q') = [q, q'],$$

is a biderivation of the crossed module $(\mathfrak{n}, \mathfrak{q}, \mu)$.

The following lemma is necessary in order to prove that $\mathrm{Bider}(\mathfrak{n}, \mathfrak{q}, \mu)$ is indeed a Leibniz algebra.

Lemma 6.3.6. *Let* $((\sigma_1, \theta_1), (\sigma_2, \theta_2)), ((\sigma_1', \theta_1'), (\sigma_2', \theta_2')) \in \mathrm{Bider}(\mathfrak{n}, \mathfrak{q}, \mu)$ *and* $(d, D) \in \mathrm{Bider}(\mathfrak{q}, \mathfrak{n})$*. Then*

$$\begin{aligned} [D\sigma_2(q), q'] &= [D\theta_2(q), q'], & [D\sigma_2(q), n] &= [D\theta_2(q), n], \\ [q, D\sigma_2(q')] &= [q, D\theta_2(q')], & [n, D\sigma_2(q)] &= [n, D\theta_2(q)], \\ [\theta_1 d(q), q'] &= [\theta_1 D(q), q'], & [\theta_1 d(q), n] &= [\theta_1 D(q), n], \\ [q, \theta_1 d(q')] &= [q, \theta_1 D(q')], & [n, \theta_1 d(q)] &= [n, \theta_1 D(q)], \\ [\theta_1\sigma_1'(n), q] &= [\theta_1\theta_1'(n), q], & [\theta_2\sigma_2'(q), n] &= [\theta_2\theta_2'(q), n], \\ [q, \theta_1\sigma_1'(n)] &= [q, \theta_1\theta_1'(n)], & [n, \theta_2\sigma_2'(q)] &= [n, \theta_2\theta_2'(q)], \end{aligned}$$

for all $n \in \mathfrak{n}$*,* $q, q' \in \mathfrak{q}$*.*

Proof. Let us show how to prove the first identity; the rest of them can be checked similarly. Let $q, q' \in \mathfrak{q}$, $(d, D) \in \mathrm{Bider}(\mathfrak{q}, \mathfrak{n})$ and $((\sigma_1, \theta_1), (\sigma_2, \theta_2)) \in \mathrm{Bider}(\mathfrak{n}, \mathfrak{q}, \mu)$. Since (σ_2, θ_2) is a biderivation of $\mathfrak{q}$, we have that $[q', \sigma_2(q)] = [q', \theta_2(q)]$. Therefore $D([q', \sigma_2(q)]) = D([q', \theta_2(q)])$. Directly from (6.3.2), we get that

$$[D(q'), \sigma_2(q)] - [D\sigma_2(q), q'] = [D(q'), \theta_2(q)] - [D\theta_2(q), q'].$$

Thus, due to (6.3.12), $[D(q'), \sigma_2(q)] = [D(q'), \theta_2(q)]$. Hence, $[D\sigma_2(q), q'] = [D\theta_2(q), q']$. □

The $\mathbf{k}$-module structure of $\mathrm{Bider}(\mathfrak{n}, \mathfrak{q}, \mu)$ is evident, while its Leibniz structure is described as follows.

Proposition 6.3.7. $\mathrm{Bider}(\mathfrak{n}, \mathfrak{q}, \mu)$ *is a Leibniz algebra with the bracket given by*

$$\begin{aligned} &[((\sigma_1, \theta_1), (\sigma_2, \theta_2)), ((\sigma_1', \theta_1'), (\sigma_2', \theta_2'))] = ([(\sigma_1, \theta_1), (\sigma_1', \theta_1')], [(\sigma_2, \theta_2), (\sigma_2', \theta_2')]) \\ &\qquad = ((\sigma_1\sigma_1' - \sigma_1'\sigma_1, \theta_1\sigma_1' - \sigma_1'\theta_1), (\sigma_2\sigma_2' - \sigma_2'\sigma_2, \theta_2\sigma_2' - \sigma_2'\theta_2)), \qquad (6.3.13) \end{aligned}$$

for all $((\sigma_1, \theta_1), (\sigma_2, \theta_2)), ((\sigma_1', \theta_1'), (\sigma_2', \theta_2')) \in \mathrm{Bider}(\mathfrak{n}, \mathfrak{q}, \mu)$*.*

Proof. It follows directly from Lemma 6.3.6. □

Proposition 6.3.8. *The* $\mathbf{k}$*-linear map* $\Delta\colon \operatorname{Bider}(\mathfrak{q},\mathfrak{n}) \to \operatorname{Bider}(\mathfrak{n},\mathfrak{q},\mu)$*, given by* $(d,D) \mapsto ((d\mu, D\mu),(\mu d, \mu D))$ *is a homomorphism of Leibniz algebras.*

Proof. Δ is well defined due to Lemma 6.3.2, while checking that it is a homomorphism of Leibniz algebras is a matter of straightforward calculations. □

Since we aspire to make Δ into a Leibniz crossed module, we need to define an action of $\operatorname{Bider}(\mathfrak{n},\mathfrak{q},\mu)$ on $\operatorname{Bider}(\mathfrak{q},\mathfrak{n})$.

Theorem 6.3.9. *There is an action of* $\operatorname{Bider}(\mathfrak{n},\mathfrak{q},\mu)$ *on* $\operatorname{Bider}(\mathfrak{q},\mathfrak{n})$ *given by:*

$$[((\sigma_1,\theta_1),(\sigma_2,\theta_2)),(d,D)] = (\sigma_1 d - d\sigma_2, \theta_1 d - d\theta_2), \tag{6.3.14}$$
$$[(d,D),((\sigma_1,\theta_1),(\sigma_2,\theta_2))] = (d\sigma_2 - \sigma_1 d, D\sigma_2 - \sigma_1 D), \tag{6.3.15}$$

for all $((\sigma_1,\theta_1),(\sigma_2,\theta_2)) \in \operatorname{Bider}(\mathfrak{n},\mathfrak{q},\mu)$*,* $(d,D) \in \operatorname{Bider}(\mathfrak{q},\mathfrak{n})$*. Moreover, the Leibniz homomorphism* Δ *(see Proposition 6.3.8) together with the above action is a Leibniz crossed module.*

Proof. Let $(d,D) \in \operatorname{Bider}(\mathfrak{q},\mathfrak{n})$ and $((\sigma_1,\theta_1),(\sigma_2,\theta_2)) \in \operatorname{Bider}(\mathfrak{n},\mathfrak{q},\mu)$. Checking that both $(\sigma_1 d - d\sigma_2, \theta_1 d - d\theta_2)$ and $(d\sigma_2 - \sigma_1 d, D\sigma_2 - \sigma_1 D)$ satisfy conditions (6.3.1) and (6.3.2) requires the combined use of the properties satisfied by the elements in $\operatorname{Bider}(\mathfrak{n},\mathfrak{q},\mu)$ and (d,D), but calculations are fairly straightforward. As an example, we show how to prove that $(\sigma_1 d - d\sigma_2, \theta_1 d - d\theta_2)$ verifies (6.3.1). Let $q,q' \in \mathfrak{q}$. Then

$$\begin{aligned}(\sigma_1 d - d\sigma_2)([q,q']) &= \sigma_1([d(q),q'] + [q,d(q')]) - d([\sigma_2(q),q'] + [q,\sigma_2(q')])\\ &= [\sigma_1 d(q),q'] + [d(q),\sigma_2(q')] + [\sigma_2(q),d(q')] + [q,\sigma_1 d(q')]\\ &\quad - [d\sigma_2(q),q'] - [\sigma_2(q),d(q')] - [d(q),\sigma_2(q')] - [q,d\sigma_2(q')]\\ &= [(\sigma_1 d - d\sigma_2)(q),q'] + [q,(\sigma_1 d - d\sigma_2)(q')].\end{aligned}$$

As for condition (6.3.3), in the case of $(\sigma_1 d - d\sigma_2, \theta_1 d - d\theta_2)$, it follows from (6.3.11), the identity (6.3.3) for (d,D) and the second identity in the first column from Lemma 6.3.6. Namely,

$$\begin{aligned}[q,(\sigma_1 d - d\sigma_2)(q')] &= [q,\sigma_1 d(q')] - [q,d\sigma_2(q')] = [q,\theta_1 d(q')] - [q,D\sigma_2(q')]\\ &= [q,\theta_1 d(q')] - [q,D\theta_2(q')] = [q,\theta_1 d(q')] - [q,d\theta_2(q')],\end{aligned}$$

for all $q, q' \in \mathfrak{q}$. A similar procedure allows to prove that $(d\sigma_2 - \sigma_1 d, D\sigma_2 - \sigma_1 D)$ satisfies condition (6.3.3) as well.

Routine calculations show that (6.3.14) and (6.3.15) together with the definition of the brackets in $\mathrm{Bider}(\mathfrak{n}, \mathfrak{q}, \mu)$ and $\mathrm{Bider}(\mathfrak{q}, \mathfrak{n})$ provide an action of Leibniz algebras.

It only remains to prove that Δ satisfies the equivariance and the Peiffer identity. It is immediate to check that

$$\Delta([((\sigma_1, \theta_1), (\sigma_2, \theta_2)), (d, D)]) = ((\sigma_1 d\mu - d\sigma_2\mu, \theta_1 d\mu - d\theta_2\mu), \\ (\mu\sigma_1 d - \mu d\sigma_2, \mu\theta_1 d - \mu d\theta_2)), \quad (6.3.16)$$

while

$$[((\sigma_1, \theta_1), (\sigma_2, \theta_2)), \Delta(d, D)] = ((\sigma_1 d\mu - d\mu\sigma_1, \theta_1 d\mu - d\mu\theta_1), \\ (\sigma_2\mu d - \mu d\sigma_2, \theta_2\mu d - \mu d\theta_2)). \quad (6.3.17)$$

Condition (6.3.6) guarantees that (6.3.16) = (6.3.17). The other identity can be checked similarly. The Peiffer identity follows immediately from (6.3.14) and (6.3.15) along the definition of Δ and the bracket in $\mathrm{Bider}(\mathfrak{q}, \mathfrak{n})$. □

6.4 The actor

In [13], Orzech introduced the notion of category of interest, which is nothing but a category of groups with operations verifying two extra conditions. **Lb** is a category of interest, although **XLb** is not. Nevertheless, it is equivalent to the category of *cat*^{1}-Leibniz algebras (see for example [6]), which is itself a modified category of interest in the sense of [2]. So it makes sense to study representability of actions in **XLb** under the context of modified categories of interest, as it is done in [2] for crossed modules of associative algebras. However, since **XLb** is an example of semi-abelian categories, and an action is the same as a split extension in any semi-abelian category [1, Lemma 1.3], we choose a different, more combinatorial approach to the problem, by constructing the semidirect product (split extension) of Leibniz crossed modules.

We use the term *actor* (as in [2, 3]) for an object which represents actions in a semi-abelian category, the general definition of which is known from [1] under the name *split extension classifier*.

We need to remark that, given a Leibniz algebra $\mathfrak{m}$, $\operatorname{Bider}(\mathfrak{m})$ is the actor of $\mathfrak{m}$ under certain conditions. In particular, the following result is proved in [3].

Proposition 6.4.1 ([3]). *Let $\mathfrak{m}$ be a Leibniz algebra such that $\operatorname{Ann}(\mathfrak{m}) = 0$ or $[\mathfrak{m}, \mathfrak{m}] = \mathfrak{m}$. Then $\operatorname{Bider}(\mathfrak{m})$ is the actor of $\mathfrak{m}$.*

Bearing in mind the ease of the generalization of the actor in the category of groups and Lie algebras to crossed modules, together with the role of $\operatorname{Bider}(\mathfrak{m})$ in regard to any Leibniz algebra $\mathfrak{m}$, it makes sense to consider $(\operatorname{Bider}(\mathfrak{q}, \mathfrak{n}), \operatorname{Bider}(\mathfrak{n}, \mathfrak{q}, \mu), \Delta)$ as a candidate for actor in **XLb**, at least under certain conditions (see Proposition 6.4.1). However, it would be reckless to define an action of a Leibniz crossed module $(\mathfrak{m}, \mathfrak{p}, \eta)$ on $(\mathfrak{n}, \mathfrak{q}, \mu)$ as a homomorphism from $(\mathfrak{m}, \mathfrak{p}, \eta)$ to the Leibniz crossed module $(\operatorname{Bider}(\mathfrak{q}, \mathfrak{n}), \operatorname{Bider}(\mathfrak{n}, \mathfrak{q}, \mu), \Delta)$, since we cannot ensure that the mentioned homomorphism induces a set of actions of $(\mathfrak{m}, \mathfrak{p}, \eta)$ on $(\mathfrak{n}, \mathfrak{q}, \mu)$ from which we can construct the semidirect product.

In [5, Proposition 2.1] the authors give an equivalent description of an action of a crossed module of groups in terms of equations. A similar description can be done for an action of a crossed module of Lie algebras (see [4]). This determines our approach to the problem. We consider a homomorphism from a Leibniz crossed module $(\mathfrak{m}, \mathfrak{p}, \eta)$ to $(\operatorname{Bider}(\mathfrak{q}, \mathfrak{n}), \operatorname{Bider}(\mathfrak{n}, \mathfrak{q}, \mu), \Delta)$, which will be denoted by $\overline{\operatorname{Act}}(\mathfrak{n}, \mathfrak{q}, \mu)$ from now on, and unravel all the properties satisfied by the mentioned homomorphism, transforming them into a set of equations. Then we check that the existence of that set of equations is equivalent to the existence of a homomorphism of Leibniz crossed modules from $(\mathfrak{m}, \mathfrak{p}, \eta)$ to $\overline{\operatorname{Act}}(\mathfrak{n}, \mathfrak{q}, \mu)$ only under certain conditions. Finally we prove that those equations indeed describe a comprehensive set of actions by constructing the associated semidirect product, which is an object in **XLb**.

Lemma 6.4.2.

(i) *Let $\mathfrak{q}$ be a Leibniz algebra and $(\sigma, \theta), (\sigma', \theta') \in \operatorname{Bider}(\mathfrak{q})$. If $\operatorname{Ann}(\mathfrak{q}) = 0$ or $[\mathfrak{q}, \mathfrak{q}] = \mathfrak{q}$, then*

$$\theta\sigma'(q) = \theta\theta'(q), \tag{6.4.1}$$

for all $q \in \mathfrak{q}$.

(ii) *Let $(\mathfrak{n}, \mathfrak{q}, \mu)$ be a Leibniz crossed module. Then we have that $((\sigma_1, \theta_1), (\sigma_2, \theta_2)) \in \operatorname{Bider}(\mathfrak{n}, \mathfrak{q}, \mu)$ and $(d, D) \in \operatorname{Bider}(\mathfrak{q}, \mathfrak{n})$. If $\operatorname{Ann}(\mathfrak{n}) =$*

0 *or* $[\mathfrak{q},\mathfrak{q}]=\mathfrak{q}$, *then*

$$D\sigma_2(q)=D\theta_2(q), \tag{6.4.2}$$
$$\theta_1 d(q)=\theta_1 D(q), \tag{6.4.3}$$

for all $q\in\mathfrak{q}$.

Proof. Calculations in order to prove (i) are straightforward. Regarding (ii), $D\sigma_2(q)-D\theta_2(q)$ and $\theta_1 d(q)-\theta_1 D(q)$ are elements in $\mathrm{Ann}(\mathfrak{n})$, immediately from the identities in the second column from Lemma 6.3.6. Therefore, if $\mathrm{Ann}(\mathfrak{n})=0$, it is clear that (6.4.2) and (6.4.3) hold.

Let us now assume that $[\mathfrak{q},\mathfrak{q}]=\mathfrak{q}$. Given $q,q'\in\mathfrak{q}$, directly from the fact that $(\sigma_2,\theta_2)\in\mathrm{Bider}(\mathfrak{q})$ and $(d,D)\in\mathrm{Bider}(\mathfrak{q},\mathfrak{n})$, we get that

$$D\theta_2([q,q'])=[D\theta_2(q),q']-[D(q'),\theta_2(q)]-[D\theta_2(q'),q]+[D(q),\theta_2(q')],$$
$$D\sigma_2([q,q'])=[D\sigma_2(q),q']-[D(q'),\sigma_2(q)]+[D(q),\sigma_2(q')]-[D\sigma_2(q'),q].$$

Due to (6.3.12) and the first identity in the first column from Lemma 6.3.6, $D\theta_2([q,q'])=D\sigma_2([q,q'])$. By hypothesis, every element in $\mathfrak{q}$ can be expressed as a linear combination of elements of the form $[q,q']$. This fact together with the linearity of D, σ_2 and θ_2, guarantees that $D\theta_2(q)=D\sigma_2(q)$ for all $q\in\mathfrak{q}$. The identity (6.4.3) can be checked similarly by making use of (6.3.3), (6.3.9), (6.3.10) and the third identity in the first column from Lemma 6.3.6. □

Theorem 6.4.3. *Let* $(\mathfrak{m},\mathfrak{p},\eta)$ *and* $(\mathfrak{n},\mathfrak{q},\mu)$ *in* ***XLb***. *There exists a homomorphism of crossed modules from* $(\mathfrak{m},\mathfrak{p},\eta)$ *to* $(\mathrm{Bider}(\mathfrak{q},\mathfrak{n}),\mathrm{Bider}(\mathfrak{n},\mathfrak{q},\mu),\Delta)$, *if the following conditions hold:*

(i) *There are actions of the Leibniz algebra* $\mathfrak{p}$ *(and so* $\mathfrak{m}$*) on the Leibniz algebras* $\mathfrak{n}$ *and* $\mathfrak{q}$. *The homomorphism* μ *is* $\mathfrak{p}$*-equivariant, that is*

$$\mu([p,n])=[p,\mu(n)], \tag{LbEQ1}$$
$$\mu([n,p])=[\mu(n),p], \tag{LbEQ2}$$

and the actions of $\mathfrak{p}$ and $\mathfrak{q}$ on $\mathfrak{n}$ are compatible, that is

$$[n,[p,q]] = [[n,p],q] - [[n,q],p], \tag{LbCOM1}$$
$$[p,[n,q]] = [[p,n],q] - [[p,q],n], \tag{LbCOM2}$$
$$[p,[q,n]] = [[p,q],n] - [[p,n],q], \tag{LbCOM3}$$
$$[n,[q,p]] = [[n,q],p] - [[n,p],q], \tag{LbCOM4}$$
$$[q,[n,p]] = [[q,n],p] - [[q,p],n], \tag{LbCOM5}$$
$$[q,[p,n]] = [[q,p],n] - [[q,n],p], \tag{LbCOM6}$$

for all $n \in \mathfrak{n}$, $p \in \mathfrak{p}$ and $q \in \mathfrak{q}$.

(i) *There are two $\mathbf{k}$-bilinear maps $\xi_1 \colon \mathfrak{m} \times \mathfrak{q} \to \mathfrak{n}$ and $\xi_2 \colon \mathfrak{q} \times \mathfrak{m} \to \mathfrak{n}$ such that*

$$\mu\xi_2(q,m) = [q,m], \tag{LbM1a}$$
$$\mu\xi_1(m,q) = [m,q], \tag{LbM1b}$$
$$\xi_2(\mu(n),m) = [n,m], \tag{LbM2a}$$
$$\xi_1(m,\mu(n)) = [m,n], \tag{LbM2b}$$
$$\xi_2(q,[p,m]) = \xi_2([q,p],m) - [\xi_2(q,m),p], \tag{LbM3a}$$
$$\xi_1([p,m],q) = \xi_2([p,q],m) - [p,\xi_2(q,m)], \tag{LbM3b}$$
$$\xi_2(q,[m,p]) = [\xi_2(q,m),p] - \xi_2([q,p],m), \tag{LbM3c}$$
$$\xi_1([m,p],q) = [\xi_1(m,q),p] - \xi_1(m,[q,p]), \tag{LbM3d}$$
$$\xi_2(q,[m,m']) = [\xi_2(q,m),m'] - [\xi_2(q,m'),m], \tag{LbM4a}$$
$$\xi_1([m,m'],q) = [\xi_1(m,q),m'] - [m,\xi_2(q,m')], \tag{LbM4b}$$
$$\xi_2([q,q'],m) = [\xi_2(q,m),q'] + [q,\xi_2(q',m)], \tag{LbM5a}$$
$$\xi_1(m,[q,q']) = [\xi_1(m,q),q'] - [\xi_1(m,q'),q], \tag{LbM5b}$$
$$[q,\xi_1(m,q')] = -[q,\xi_2(q',m)], \tag{LbM5c}$$
$$\xi_1(m,[p,q]) = -\xi_1(m,[q,p]), \tag{LbM6a}$$
$$[p,\xi_1(m,q)] = -[p,\xi_2(q,m)], \tag{LbM6b}$$

for all $m,m' \in \mathfrak{m}$, $n \in \mathfrak{n}$, $p \in \mathfrak{p}$, $q,q' \in \mathfrak{q}$.

Additionally, the converse statement is also true if one of the following *conditions holds:*

$$\mathrm{Ann}(\mathfrak{n}) = 0 = \mathrm{Ann}(\mathfrak{q}), \tag{CON1}$$

$$\mathrm{Ann}(\mathfrak{n}) = 0 \quad \textit{and} \quad [\mathfrak{q}, \mathfrak{q}] = \mathfrak{q}, \tag{CON2}$$

$$[\mathfrak{n}, \mathfrak{n}] = \mathfrak{n} \quad \textit{and} \quad [\mathfrak{q}, \mathfrak{q}] = \mathfrak{q}. \tag{CON3}$$

Proof. Let us suppose that (i) and (ii) hold. It is possible to define a homomorphism of crossed modules (φ, ψ) from $(\mathfrak{m}, \mathfrak{p}, \eta)$ to $\overline{\mathrm{Act}}(\mathfrak{n}, \mathfrak{q}, \mu)$ as follows. Given $m \in \mathfrak{m}$, $\varphi(m) = (d_m, D_m)$, with

$$d_m(q) = -\xi_2(q, m), \qquad D_m(q) = \xi_1(m, q),$$

for all $q \in \mathfrak{q}$. On the other hand, for any $p \in \mathfrak{p}$, $\psi(p) = ((\sigma_1^p, \theta_1^p), (\sigma_2^p, \theta_2^p))$, with

$$\begin{aligned} \sigma_1^p(n) &= -[n, p], \qquad & \theta_1^p(n) &= [p, n], \\ \sigma_2^p(q) &= -[q, p], \qquad & \theta_2^p(q) &= [p, q], \end{aligned}$$

for all $n \in \mathfrak{n}$, $q \in \mathfrak{q}$. It follows directly from (LbM5a)–(LbM5c) that $(d_m, D_m) \in \mathrm{Bider}(\mathfrak{q}, \mathfrak{n})$ for all $m \in \mathfrak{m}$. Besides, φ is clearly $\mathbf{k}$-linear and given $m, m' \in \mathfrak{m}$,

$$[\varphi(m), \varphi(m')] = [(d_m, D_m), (d_{m'}, D_{m'})] = [d_m \mu d_{m'} - d_{m'} \mu d_m, D_m \mu d_{m'} - d_{m'} \mu D_m].$$

For any $q \in \mathfrak{q}$,

$$\begin{aligned} d_m \mu d_{m'}(q) - d_{m'} \mu d_m(q) &= -\xi_2(\mu d_{m'}(q), m) + \xi_2(\mu d_m(q), m') \\ &= -[d_{m'}(q), m] + [d_m(q), m'] \\ &= [\xi_2(q, m'), m] - [\xi_2(q, m), m'] \\ &= -\xi_2(q, [m, m']) = d_{[m,m']}(q), \end{aligned}$$

due to (LbM2a) and (LbM4a). Analogously, it can be easily checked the identity $(D_m \mu d_{m'} - d_{m'} \mu D_m)(q) = D_{[m,m']}(q)$ by making use of (LbM2a), (LbM2b) and (LbM4b). Hence, φ is a homomorphism of Leibniz algebras.

As for ψ, it is necessary to prove that $((\sigma_1^p, \theta_1^p), (\sigma_2^p, \theta_2^p))$ satisfies all the axioms from Definition 6.3.5 for any $p \in \mathfrak{p}$. The fact that (σ_1^p, θ_1^p) (respectively (σ_2^p, θ_2^p)) is a biderivation of $\mathfrak{n}$ (respectively $\mathfrak{q}$) follows directly from the actions of $\mathfrak{p}$ on $\mathfrak{n}$ and $\mathfrak{q}$. The identities $\mu \theta_1^p = \theta_2^p \mu$ and $\mu \sigma_1^p = \sigma_2^p \mu$ are immediate consequences of (LbEQ1) and (LbEQ2) respectively.

Observe that the combinations of the identities (LbCOM1) and (LbCOM4) and the identities (LbCOM5) and (LbCOM6) yield the equalities

$$-[n,[q,p]] = [n,[p,q]] \qquad \text{and} \qquad -[q,[n,p]] = [q,[p,n]].$$

These together with (LbCOM2)–(LbCOM5) allow us to prove that $((\sigma_1^p, \theta_1^p), (\sigma_2^p, \theta_2^p))$ does satisfy conditions (6.3.7)–(6.3.12) from Definition 6.3.5. Therefore, ψ is well defined, while it is obviously $\mathbf{k}$-linear. Moreover, due to (6.3.13) we know that

$$[\psi(p),\psi(p')] = ((\sigma_1^p\sigma_1^{p'} - \sigma_1^{p'}\sigma_1^p, \theta_1^p\sigma_1^{p'} - \sigma_1^{p'}\theta_1^p), (\sigma_2^p\sigma_2^{p'} - \sigma_2^{p'}\sigma_2^p, \theta_2^p\sigma_2^{p'} - \sigma_2^{p'}\theta_2^p)),$$

and by definition

$$\psi([p,p']) = ((\sigma_1^{[p,p']}, \theta_1^{[p,p']}), (\sigma_2^{[p,p']}, \theta_2^{[p,p']})).$$

One can easily check that the corresponding components are equal by making use of the actions of $\mathfrak{p}$ on $\mathfrak{n}$ and $\mathfrak{q}$. Hence, ψ is a homomorphism of Leibniz algebras.

Recall that

$$\begin{aligned}\Delta\varphi(m) &= ((d_m\mu, D_m\mu), (\mu d_m, \mu D_m)),\\ \psi\eta(m) &= ((\sigma_1^{\eta(m)}, \theta_1^{\eta(m)}), (\sigma_2^{\eta(m)}, \theta_2^{\eta(m)})),\end{aligned}$$

for any $m \in \mathfrak{m}$, but

$$\begin{aligned}d_m\mu(n) &= -\xi_2(\mu(n),m) = -[n,m] = -[n,\eta(m)] = \sigma_1^{\eta(m)}(n),\\ D_m\mu(n) &= \xi_1(m,\mu(n)) = [m,n] = [\eta(m),n] = \theta_1^{\eta(m)}(n),\\ \mu d_m(q) &= -\mu\xi_2(q,m) = -[q,m] = -[q,\eta(m)] = \sigma_2^{\eta(m)}(q),\\ \mu D_m(q) &= \mu\xi_1(m,q) = [m,q] = [\eta(m),q] = \theta_2^{\eta(m)}(q),\end{aligned}$$

for all $n \in \mathfrak{n}$, $q \in \mathfrak{q}$, due to (LbM1a), (LbM1b), (LbM2a), (LbM2b). Therefore, $\Delta\varphi = \psi\eta$.

It only remains to check the behaviour of (φ,ψ) regarding the action of $\mathfrak{p}$ on $\mathfrak{m}$. Let $m \in \mathfrak{m}$ and $p \in \mathfrak{p}$. Due to (6.3.14) and (6.3.15),

$$\begin{aligned}[\psi(p),\varphi(m)] &= (\sigma_1^p d_m - d_m\sigma_2^p, \theta_1^p d_m - d_m\theta_2^p),\\ [\varphi(m),\psi(p)] &= (d_m\sigma_2^p - \sigma_1^p d_m, D_m\sigma_2^p - \sigma_1^p D_m).\end{aligned}$$

On the other hand, by definition, we know that

$$\varphi([p,m]) = (d_{[p,m]}, D_{[p,m]}),$$
$$\varphi([m,p]) = (d_{[m,p]}, D_{[m,p]}).$$

Directly from (LbM3a), (LbM3b), (LbM3c) and (LbM3d) one can easily confirm that the required identities between components hold. Hence, we can finally ensure that (φ, ψ) is a homomorphism of Leibniz crossed modules.

Now let us show that it is necessary that at least one of the conditions (CON1)–(CON3) holds in order to prove the converse statement. Let us suppose that there is a homomorphism of crossed modules

$$\begin{array}{ccc} \mathfrak{m} & \xrightarrow{\eta} & \mathfrak{p} \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ \operatorname{Bider}(\mathfrak{q},\mathfrak{n}) & \xrightarrow[\Delta]{} & \operatorname{Bider}(\mathfrak{n},\mathfrak{q},\mu) \end{array} \qquad (6.4.4)$$

Given $m \in \mathfrak{m}$ and $p \in \mathfrak{p}$, let us denote $\varphi(m)$ by (d_m, D_m) and $\psi(p)$ by $((\sigma_1^p, \theta_1^p), (\sigma_2^p, \theta_2^p))$, which satisfy conditions (6.3.1)–(6.3.3) from Definition 6.3.1 and conditions (6.3.5)–(6.3.12) from Definition 6.3.5 respectively. Also, due to the definition of Δ (see Proposition 6.3.8), the commutativity of (6.4.4) can be expressed by the identity

$$((d_m\mu, D_m\mu), (\mu d_m, \mu D_m)) = ((\sigma_1^{\eta(m)}, \theta_1^{\eta(m)}), (\sigma_2^{\eta(m)}, \theta_2^{\eta(m)})), \qquad (6.4.5)$$

for all $m \in \mathfrak{m}$. It is possible to define four bilinear maps, all of them denoted by $[-,-]$, from $\mathfrak{p} \times \mathfrak{n}$ to $\mathfrak{n}$, $\mathfrak{n} \times \mathfrak{p}$ to $\mathfrak{n}$, $\mathfrak{p} \times \mathfrak{q}$ to $\mathfrak{q}$ and $\mathfrak{q} \times \mathfrak{p}$ to $\mathfrak{q}$, given by

$$[p,n] = \theta_1^p(n), \qquad [n,p] = -\sigma_1^p(n),$$
$$[p,q] = \theta_2^p(q), \qquad [q,p] = -\sigma_2^p(q),$$

for all $n \in \mathfrak{n}$, $p \in \mathfrak{p}$, $q \in \mathfrak{q}$. These maps define actions of $\mathfrak{p}$ on $\mathfrak{n}$ and $\mathfrak{q}$. The first three identities for the action on $\mathfrak{n}$ (respectively $\mathfrak{q}$) follow easily from the fact that (σ_1^p, θ_1^p) (respectively (σ_2^p, θ_2^p)) is a biderivation of $\mathfrak{n}$ (respectively $\mathfrak{q}$).

Since ψ is a Leibniz homomorphism, we get that

$$((\sigma_1^{[p,p']}, \theta_1^{[p,p']}), (\sigma_2^{[p,p']}, \theta_2^{[p,p']})) = ((\sigma_1^p\sigma_1^{p'} - \sigma_1^{p'}\sigma_1^p, \theta_1^p\sigma_1^{p'} - \sigma_1^{p'}\theta_1^p),$$
$$(\sigma_2^p\sigma_2^{p'} - \sigma_2^{p'}\sigma_2^p, \theta_2^p\sigma_2^{p'} - \sigma_2^{p'}\theta_2^p)).$$

The identities between the first and the second (respectively the third and the fourth) components in those quadruples allow us to confirm the fourth and fifth identities for the action of $\mathfrak{p}$ on $\mathfrak{n}$ (respectively $\mathfrak{q}$).

As for the last condition for both actions, it is fairly straightforward to check that

$$[[p,p'],n] - [[p,n],p'] = \theta_1^p \sigma_1^{p'}(n),$$
$$[[p,p'],q] - [[p,q],p'] = \theta_2^p \sigma_2^{p'}(q),$$

while

$$[p,[p',n]] = \theta_1^p \theta_1^{p'}(n),$$
$$[p,[p',q]] = \theta_2^p \theta_2^{p'}(q),$$

for all $n \in \mathfrak{n}$, $p, p' \in \mathfrak{p}$, $q \in \mathfrak{q}$. However, if at least one of the conditions (CON1)–(CON3) holds, due to Lemma 6.4.2 (i), $\theta_1^p \sigma_1^{p'}(n) = \theta_1^p \theta_1^{p'}(n)$ and $\theta_2^p \sigma_2^{p'}(q) = \theta_2^p \theta_2^{p'}(q)$. Therefore, we can ensure that there are Leibniz actions of $\mathfrak{p}$ on both $\mathfrak{n}$ and $\mathfrak{q}$, which induce actions of $\mathfrak{m}$ on $\mathfrak{n}$ and $\mathfrak{q}$ via η.

The reader might have noticed that a fourth possible condition on $(\mathfrak{n}, \mathfrak{q}, \mu)$ could have been considered in order to guarantee the existence of the actions of $\mathfrak{p}$ on $\mathfrak{n}$ and $\mathfrak{q}$ from the existence of the homomorphism of Leibniz crossed modules (φ, ψ). In fact, if $[\mathfrak{n}, \mathfrak{n}] = \mathfrak{n}$ and $\operatorname{Ann}(\mathfrak{q}) = 0$, the problem with the last condition for the actions could have been solved in the same way. Nevertheless, this fourth condition does not guarantee that (ii) holds, as we will prove immediately below.

Regarding (LbEQ1) and (LbEQ2), they follow directly from (6.3.6) (observe that, by hypothesis, $((\sigma_1^p, \theta_1^p), (\sigma_2^p, \theta_2^p))$ is a biderivation of $(\mathfrak{n}, \mathfrak{q}, \mu)$ for any $p \in \mathfrak{p}$). Similarly, (LbCOM1)–(LbCOM6) follow almost immediately from (6.3.7)–(6.3.12). Hence, (i) holds.

Concerning (ii), we can define $\xi_1(m,q) = D_m(q)$ and $\xi_2(q,m) = -d_m(q)$ for any $m \in \mathfrak{m}$, $q \in \mathfrak{q}$. In this way, ξ_1 and ξ_2 are clearly bilinear. (LbM1a), (LbM1b), (LbM2a) and (LbM2b) follow immediately from the identity (6.4.5) and the fact that the actions of $\mathfrak{m}$ on $\mathfrak{n}$ and $\mathfrak{q}$ are induced by the actions of $\mathfrak{p}$ via η.

Identities (LbM5a), (LbM5b) and (LbM5c) are a direct consequence of (6.3.1)–(6.3.3) (recall that, by hypothesis, (d_m, D_m) is a biderivation from $\mathfrak{q}$ to $\mathfrak{n}$ for any $m \in \mathfrak{m}$).

Since φ is a Leibniz homomorphism, we have that

$$(d_{[m,m']}, D_{[m,m']}) = (d_m \mu d_{m'} - d_{m'} \mu d_m, D_m \mu d_{m'} - d_{m'} \mu D_m).$$

This identity, together with (LbM2a) and (LbM2b), allows to easily prove that (LbM4a) and (LbM4b) hold.

Note that, since (φ, ψ) is a homomorphism of Leibniz crossed modules, $\varphi([p,m]) = [\psi(p), \varphi(m)]$ and $\varphi([m,p]) = [\varphi(m), \psi(p)]$ for all $m \in \mathfrak{m}$, $p \in \mathfrak{p}$. Due to the definition of the action of $\mathrm{Bider}(\mathfrak{n}, \mathfrak{q}, \mu)$ on $\mathrm{Bider}(\mathfrak{q}, \mathfrak{n})$ (see Theorem 6.3.9), we can write

$$\begin{aligned} (d_{[p,m]}, D_{[p,m]}) &= (\sigma_1^p d_m - d_m \sigma_2^p, \theta_1^p d_m - d_m \theta_2^p), \\ (d_{[m,p]}, D_{[m,p]}) &= (d_m \sigma_2^p - \sigma_1^p d_m, D_m \sigma_2^p - \sigma_1^p D_m). \end{aligned}$$

Identities (LbM3a), (LbM3b), (LbM3c) and (LbM3d) follow immediately from the previous identities.

Regarding (LbM6a) and (LbM6b), directly from the definition of ξ_1, ξ_2 and the actions of $\mathfrak{p}$ on $\mathfrak{n}$ and $\mathfrak{q}$, we have that

$$\begin{aligned} \xi_1(m, [p,q]) &= D_m \theta_2^p(q), & [p, \xi_1(m,q)] &= \theta_1^p D_m(q), \\ -\xi_1(m, [q,p]) &= D_m \sigma_2^p(q), & -[p, \xi_2(q,m)] &= \theta_1^p d_m(q), \end{aligned}$$

for all $m \in \mathfrak{m}$, $p \in \mathfrak{p}$, $q \in \mathfrak{q}$. Nevertheless, if at least one of the conditions (CON1)–(CON3) holds, due to Lemma 6.4.2 (ii), $D_m \theta_2^p(q) = D_m \sigma_2^p(q)$ and $\theta_1^p D_m(q) = \theta_1^p d_m(q)$. Hence, (ii) holds. □

Remark 6.4.4. A closer look at the proof of the previous theorem shows that neither conditions (LbM6a) and (LbM6b), nor the identities $[p,[p',n]] = [[p,p'],n] - [[p,n],p']$ and $[p,[p',q]] = [[p,p'],q] - [[p,q],p']$ (which correspond to the sixth axiom satisfied by the actions of $\mathfrak{p}$ on $\mathfrak{n}$ and $\mathfrak{q}$ respectively) are necessary in order to prove the existence of a homomorphism of crossed modules (φ, ψ) from $(\mathfrak{m}, \mathfrak{p}, \eta)$ to $\overline{\mathrm{Act}}(\mathfrak{n}, \mathfrak{q}, \mu)$, under the hypothesis that (i) and (ii) hold. Actually, if we remove those conditions from (i) and (ii), the converse statement would be true for any Leibniz crossed module $(\mathfrak{n}, \mathfrak{q}, \mu)$, even if it does not satisfy any of the conditions (CON1)–(CON3). The problem is that (LbM6a) and (LbM6b), together with the sixth identity satisfied by the actions of $\mathfrak{p}$ on $\mathfrak{n}$ and $\mathfrak{q}$ are essential in order to prove that (i) and (ii) as in Theorem 6.4.3 describe a set of actions of $(\mathfrak{m}, \mathfrak{p}, \eta)$ on $(\mathfrak{n}, \mathfrak{q}, \mu)$, as we will show immediately below. This agrees with the idea of $\overline{\mathrm{Act}}(\mathfrak{n}, \mathfrak{q}, \mu)$ not being "good enough" to be the actor of $(\mathfrak{n}, \mathfrak{q}, \mu)$ in general, just as $\mathrm{Bider}(\mathfrak{m})$ is not always the actor of a Leibniz algebra $\mathfrak{m}$.

Example 6.4.5. Let $(\mathfrak{m}, \mathfrak{p}, \eta) \in$ **XLb**, then there is a homomorphism $(\varphi, \psi) \colon (\mathfrak{m}, \mathfrak{p}, \eta) \to \overline{\mathrm{Act}}(\mathfrak{m}, \mathfrak{p}, \eta)$, with $\varphi(m) = (d_m, D_m)$ and $\psi(p) = ((\sigma_1^p, \theta_1^p), (\sigma_2^p, \theta_2^p))$, where

$$d_m(p) = -[p, m], \qquad D_m(p) = [m, p],$$

and

$$\begin{aligned} \sigma_1^p(m) &= -[m, p], & \theta_1^p(m) &= [p, m], \\ \sigma_2^p(p') &= -[p', p], & \theta_2^p(p') &= [p, p'], \end{aligned}$$

for all $m \in \mathfrak{m}$, $p, p' \in \mathfrak{p}$. Calculations in order to prove that (φ, ψ) is indeed a homomorphism of Leibniz crossed modules are fairly straightforward. Of course, this homomorphism does not necessarily define a set of actions from which it is possible to construct the semidirect product. Theorem 6.4.3, along with the result immediately bellow, shows that if $(\mathfrak{m}, \mathfrak{p}, \eta)$ satisfies at least one of the conditions (CON1)–(CON3), then the previous homomorphism does define an appropriate set of actions of $(\mathfrak{m}, \mathfrak{p}, \eta)$ on itself.

Let $(\mathfrak{m}, \mathfrak{p}, \eta)$ and $(\mathfrak{n}, \mathfrak{q}, \mu)$ be Leibniz crossed modules such that (i) and (ii) from Theorem 6.4.3 hold. Therefore, there are Leibniz actions of $\mathfrak{m}$ on $\mathfrak{n}$ and of $\mathfrak{p}$ on $\mathfrak{q}$, so it makes sense to consider the semidirect products of Leibniz algebras $\mathfrak{n} \rtimes \mathfrak{m}$ and $\mathfrak{q} \rtimes \mathfrak{p}$. Furthermore, we have the following result.

Theorem 6.4.6. *There is an action of the Leibniz algebra $\mathfrak{q} \rtimes \mathfrak{p}$ on the Leibniz algebra $\mathfrak{n} \rtimes \mathfrak{m}$, given by*

$$[(q, p), (n, m)] = ([q, n] + [p, n] + \xi_2(q, m), [p, m]), \tag{6.4.6}$$

$$[(n, m), (q, p)] = ([n, q] + [n, p] + \xi_1(m, q), [m, p]), \tag{6.4.7}$$

for all $(q, p) \in \mathfrak{q} \rtimes \mathfrak{p}$, $(n, m) \in \mathfrak{n} \rtimes \mathfrak{m}$, with ξ_1 and ξ_2 as in Theorem 6.4.3. Moreover, the Leibniz homomorphism $(\mu, \eta) \colon \mathfrak{n} \rtimes \mathfrak{m} \to \mathfrak{q} \rtimes \mathfrak{p}$, given by

$$(\mu, \eta)(n, m) = (\mu(n), \eta(m)),$$

for all $(n, m) \in \mathfrak{n} \rtimes \mathfrak{m}$, together with the previous action, is a Leibniz crossed module.

Proof. Identities (6.4.6) and (6.4.7) follow easily from the conditions satisfied by $(\mathfrak{m}, \mathfrak{p}, \eta)$ and $(\mathfrak{n}, \mathfrak{q}, \mu)$ (see Theorem 6.4.3). Nevertheless, as an example, we show how to prove the third one. Calculations for the rest of the identities are similar. Let $(n, m), (n', m') \in \mathfrak{n} \rtimes \mathfrak{m}$ and $(q, p) \in \mathfrak{q} \rtimes \mathfrak{p}$. By routine calculations we get that

$$\begin{aligned}
[(n,m),[(n',m'),(q,p)]] = (&\underbrace{[n,[n',q]]}_{(1)} \underbrace{+[n,[n',p]]}_{(2)} \underbrace{+[n,\xi_1(m',q)]}_{(3)} \underbrace{+[m,[n',q]]}_{(4)} \\
&\underbrace{+[m,[n',p]]}_{(5)} \underbrace{+[m,\xi_1(m',q)]}_{(6)} \underbrace{+[n,[m',p]]}_{(7)}, \underbrace{[m,[m',p]]}_{(8)}), \\
[[(n,m),(n',m')],(q,p)] = (&\underbrace{[[n,n'],q]}_{(1')} \underbrace{+[[n,n'],p]}_{(2')} \underbrace{+[[n,m'],q]}_{(3')} \underbrace{+[[m,n'],q]}_{(4')} \\
&\underbrace{+[[m,n'],p]}_{(5')} \underbrace{+\xi_1([m,m'],q)}_{(6')} \underbrace{+[[n,m'],p]}_{(7')}, \underbrace{[[m,m'],p]}_{(8')}), \\
[[(n,m),(q,p)],(n',m')] = (&\underbrace{[[n,q],n']}_{(1'')} \underbrace{+[[n,p],n']}_{(2'')} \underbrace{+[[n,q],m']}_{(3'')} \underbrace{+[\xi_1(m,q),n']}_{(4'')} \\
&\underbrace{+[[m,p],n']}_{(5'')} \underbrace{+[\xi_1(m,q),m']}_{(6'')} \underbrace{+[[n,p],m']}_{(7'')}, \underbrace{[[m,p],m']}_{(8'')}).
\end{aligned}$$

Let us show that $(i) = (i') - (i'')$ for $i = 1, \ldots, 8$. It is immediate for $i = 1, 2, 8$ due to the action of $\mathfrak{q}$ on $\mathfrak{n}$ and the actions of $\mathfrak{p}$ on $\mathfrak{n}$ and $\mathfrak{m}$. For $i = 5$, the identity follows from the fact that the action of $\mathfrak{m}$ on $\mathfrak{n}$ is defined via η together with the equivariance of η. Namely,

$$\begin{aligned}
[m,[n',p]] &= [\eta(m),[n',p]] = [[\eta(m),n'],p] - [[\eta(m),p],n'] \\
&= [[m,n'],p] - [\eta([m,p]),n'] = [[m,n'],p] - [[m,p],n'].
\end{aligned}$$

The procedure is similar for $i = 7$. For $i = 3$, it is necessary to make use of the Peiffer identity of μ, (LbM1b), the definition of the action of $\mathfrak{m}$ on $\mathfrak{n}$ and $\mathfrak{q}$ via η and (LbCOM1):

$$\begin{aligned}
[n,\xi_1(m',q)] &= [n,\mu\xi_1(m',q)] = [n,[m',q]] = [n,[\eta(m'),q]] \\
&= [[n,\eta(m')],q] - [[n,q],\eta(m')] = [[n,m'],q] - [[n,q],m'].
\end{aligned}$$

The conditions required in order to prove the identity for $i = 4$ are the same used for $i = 3$ except (LbCOM1), which is replaced by (LbCOM2).

Finally, for $i = 6$, due to (LbM4b) and the definition of the action of $\mathfrak{m}$ on $\mathfrak{n}$ via η, we know that

$$\xi_1([m, m'], q) = [\xi_1(m, q), m'] - [m, \xi_2(q, m')] = [\xi_1(m, q), m'] - [\eta(m), \xi_2(q, m')],$$

but applying (LbM6b), we get

$$\xi_1([m, m'], q) = [\xi_1(m, q), m'] + [\eta(m), \xi_1(m', q)] = [\xi_1(m, q), m'] + [m, \xi_1(m', q)],$$

so $(6) = (6') - (6'')$ and the third identity holds. Note that (LbM6a) and (LbM6b) are necessary in order to check the fourth and fifth identities respectively.

Checking that (μ, η) is indeed a Leibniz homomorphism follows directly from the definition of the action of $\mathfrak{m}$ on $\mathfrak{n}$ via η together with the conditions (LbEQ1) and (LbEQ2). Regarding the equivariance of (μ, η), given $(n, m) \in \mathfrak{n} \rtimes \mathfrak{m}$ and $(q, p) \in \mathfrak{q} \rtimes \mathfrak{p}$,

$$\begin{aligned}
(\mu, \eta)([(q, p), (n, m)]) &= (\mu, \eta)([q, n] + [p, n] + \xi_2(q, m), [p, m]) \\
&= (\mu([q, n]) + \mu([p, n]) + \mu\xi_2(q, m), \eta([p, m])) \\
&= ([q, \mu(n)] + [p, \mu(n)] + [q, m], [p, \eta(m)]) \\
&= ([q, \mu(n)] + [p, \mu(n)] + [q, \eta(m)], [p, \eta(m)]) \\
&= [(q, p), (\mu(n), \eta(m))],
\end{aligned}$$

due to the equivariance of μ and η, (LbEQ1), (LbM1a) and the definition of the action of $\mathfrak{m}$ on $\mathfrak{q}$ via η. Similarly, but using (LbEQ2) and (LbM1b) instead of (LbEQ1) and (LbM1a), it can be proved that $(\mu, \eta)([(n, m), (q, p)]) = [(\mu(n), \eta(m)), (q, p)]$.

The Peiffer identity of (μ, η) follows easily from the homonymous property of μ and η, the definition of the action of $\mathfrak{m}$ on $\mathfrak{n}$ via η and the conditions (LbM2a) and (LbM2b). □

Definition 6.4.7. The Leibniz crossed module $(\mathfrak{n} \rtimes \mathfrak{m}, \mathfrak{q} \rtimes \mathfrak{p}, (\mu, \eta))$ is called the *semidirect product* of the Leibniz crossed modules $(\mathfrak{n}, \mathfrak{q}, \mu)$ and $(\mathfrak{m}, \mathfrak{p}, \eta)$.

Note that the semidirect product determines an obvious split extension of $(\mathfrak{m}, \mathfrak{p}, \eta)$ by $(\mathfrak{n}, \mathfrak{q}, \mu)$

$$(0,0,0) \longrightarrow (\mathfrak{n}, \mathfrak{q}, \mu) \longrightarrow (\mathfrak{n} \rtimes \mathfrak{m}, \mathfrak{q} \rtimes \mathfrak{p}, (\mu, \eta)) \rightleftarrows (\mathfrak{m}, \mathfrak{p}, \eta) \longrightarrow (0,0,0)$$

Conversely, any split extension of $(\mathfrak{m}, \mathfrak{p}, \eta)$ by $(\mathfrak{n}, \mathfrak{q}, \mu)$ is isomorphic to their semidirect product, where the action of $(\mathfrak{m}, \mathfrak{p}, \eta)$ on $(\mathfrak{n}, \mathfrak{q}, \mu)$ is induced by the splitting homomorphism.

Remark 6.4.8. If $(\mathfrak{m}, \mathfrak{p}, \eta)$ and $(\mathfrak{n}, \mathfrak{q}, \mu)$ are Leibniz crossed modules and at least one of the following conditions holds,

1. $\mathrm{Ann}(\mathfrak{n}) = 0 = \mathrm{Ann}(\mathfrak{q})$,
2. $\mathrm{Ann}(\mathfrak{n}) = 0$ and $[\mathfrak{q}, \mathfrak{q}] = \mathfrak{q}$,
3. $[\mathfrak{n}, \mathfrak{n}] = \mathfrak{n}$ and $[\mathfrak{q}, \mathfrak{q}] = \mathfrak{q}$,

an action of the crossed module $(\mathfrak{m}, \mathfrak{p}, \eta)$ on $(\mathfrak{n}, \mathfrak{q}, \mu)$ can be also defined as a homomorphism of Leibniz crossed modules from $(\mathfrak{m}, \mathfrak{p}, \eta)$ to $\overline{\mathrm{Act}}(\mathfrak{n}, \mathfrak{q}, \mu)$. In other words, under one of those conditions, $\overline{\mathrm{Act}}(\mathfrak{n}, \mathfrak{q}, \mu)$ is the actor of $(\mathfrak{n}, \mathfrak{q}, \mu)$ and it can be denoted simply by $\mathrm{Act}(\mathfrak{n}, \mathfrak{q}, \mu)$.

Example 6.4.9.
(i) Let $\mathfrak{n}$ be an ideal of a Leibniz algebra $\mathfrak{q}$ and consider the crossed module $(\mathfrak{n}, \mathfrak{q}, \iota)$, where ι is the inclusion. It is easy to check that $\overline{\mathrm{Act}}(\mathfrak{n}, \mathfrak{q}, \iota) = (X, Y, \iota)$, where X is a Leibniz algebra isomorphic to $\{(d, D) \in \mathrm{Bider}(\mathfrak{q}) \mid d(q), D(q) \in \mathfrak{n} \text{ for all } q \in \mathfrak{q}\}$ and Y is a Leibniz algebra isomorphic to $\{(d, D) \in \mathrm{Bider}(\mathfrak{q}) \mid (d_{|\mathfrak{n}}, D_{|\mathfrak{n}}) \in \mathrm{Bider}(\mathfrak{n})\}$.
(ii) Given a Leibniz algebra $\mathfrak{q}$, it can be regarded as a Leibniz crossed module in two obvious ways, $(0, \mathfrak{q}, 0)$ and $(\mathfrak{q}, \mathfrak{q}, \mathrm{id}_{\mathfrak{q}})$. As a particular case of the previous example, one can easily check that $\overline{\mathrm{Act}}(0, \mathfrak{q}, 0) \cong (0, \mathrm{Bider}(\mathfrak{q}), 0)$ and $\overline{\mathrm{Act}}(\mathfrak{q}, \mathfrak{q}, \mathrm{id}_{\mathfrak{q}}) \cong (\mathrm{Bider}(\mathfrak{q}), \mathrm{Bider}(\mathfrak{q}), \mathrm{id})$.
(iii) Every Lie crossed module $(\mathbb{n}, \mathbb{q}, \mu)$ can be regarded as a Leibniz crossed module (see for instance [6, Remark 3.9]). Note that in this situation, both the multiplication and the action are antisymmetric. The actor of $(\mathbb{n}, \mathbb{q}, \mu)$ is $(\mathrm{Der}(\mathbb{q}, \mathbb{n}), \mathrm{Der}(\mathbb{n}, \mathbb{q}, \mu), \Delta)$, where $\mathrm{Der}(\mathbb{q}, \mathbb{n})$ is the Lie algebra of all derivations from $\mathbb{q}$ to $\mathbb{n}$ and $\mathrm{Der}(\mathbb{n}, \mathbb{q}, \mu)$ is the Lie algebra of derivations of the crossed module $(\mathbb{n}, \mathbb{q}, \mu)$ (see [7] for the details). Given $(d, D) \in \mathrm{Bider}(\mathbb{q}, \mathbb{n})$, both d and D are elements in $\mathrm{Der}(\mathbb{q}, \mathbb{n})$. Additionally, if we assume that at least one of the conditions from the previous lemma holds, then either $\mathrm{Ann}(\mathbb{n}) = 0$ or $[\mathbb{q}, \mathbb{q}] = \mathbb{q}$. In this situation, one can easily derive from (6.3.3) that $\mathrm{Bider}(\mathbb{q}, \mathbb{n}) = \{(d, d) \mid d \in \mathrm{Der}(\mathbb{q}, \mathbb{n})\}$. Besides, the bracket in $\mathrm{Bider}(\mathbb{q}, \mathbb{n})$ becomes antisymmetric and, as a Lie algebra, it is isomorphic to $\mathrm{Der}(\mathbb{q}, \mathbb{n})$. Similarly, $\mathrm{Bider}(\mathbb{n}, \mathbb{q}, \mu)$ is a Lie algebra isomorphic to $\mathrm{Der}(\mathbb{n}, \mathbb{q}, \mu)$ and $\overline{\mathrm{Act}}(\mathbb{n}, \mathbb{q}, \mu)$ is a Lie crossed module isomorphic to $\mathrm{Act}(\mathbb{n}, \mathbb{q}, \mu)$.

6.5 Center of a Leibniz crossed module

Let us assume in this section that $(\mathfrak{n}, \mathfrak{q}, \mu)$ is a Leibniz crossed module that satisfies at least one of the conditions (CON1)–(CON3). Denote by $\mathrm{Z}(\mathfrak{q})$ the center of the Leibniz algebra $\mathfrak{q}$, which in this case coincides with its annihilator (note that the center and the annihilator are not the same object in general). Consider the canonical homomorphism (φ, ψ) from $(\mathfrak{n}, \mathfrak{q}, \mu)$ to $\mathrm{Act}(\mathfrak{n}, \mathfrak{q}, \mu)$, as in Example 6.4.5. It is easy to check that

$$\mathrm{Ker}(\varphi) = \mathfrak{n}^{\mathfrak{q}} \quad \text{and} \quad \mathrm{Ker}(\psi) = \mathrm{st}_{\mathfrak{q}}(\mathfrak{n}) \cap \mathrm{Z}(\mathfrak{q}),$$

where $\mathfrak{n}^{\mathfrak{q}} = \{n \in \mathfrak{n} \mid [q, n] = [n, q] = 0, \text{ for all } q \in \mathfrak{q}\}$ and $\mathrm{st}_{\mathfrak{q}}(\mathfrak{n}) = \{q \in \mathfrak{q} \mid [q, n] = [n, q] = 0, \text{ for all } n \in \mathfrak{n}\}$. Therefore, the kernel of (φ, ψ) is the Leibniz crossed module $(\mathfrak{n}^{\mathfrak{q}}, \mathrm{st}_{\mathfrak{q}}(\mathfrak{n}) \cap \mathrm{Z}(\mathfrak{q}), \mu)$. Thus, the kernel of (φ, ψ) coincides with the center of the crossed module $(\mathfrak{n}, \mathfrak{q}, \mu)$, as defined in the preliminary version of [14, Definition 27] for crossed modules in modified categories of interest. This definition of center agrees with the categorical notion of center by Huq [8] and confirms that our construction of the actor for a Leibniz crossed module is consistent.

Example 6.5.1. Consider the crossed module $(\mathfrak{n}, \mathfrak{q}, \iota)$, where $\mathfrak{n}$ is an ideal of $\mathfrak{q}$ and ι is the inclusion. Then, its center is given by the Leibniz crossed module $(\mathfrak{n} \cap \mathrm{Z}(\mathfrak{q}), \mathrm{Z}(\mathfrak{q}), \iota)$. In particular, the center of $(0, \mathfrak{q}, 0)$ is $(0, \mathrm{Z}(\mathfrak{q}), 0)$ and the center of $(\mathfrak{q}, \mathfrak{q}, \mathrm{id}_{\mathfrak{q}})$ is $(\mathrm{Z}(\mathfrak{q}), \mathrm{Z}(\mathfrak{q}), \mathrm{id})$.

By analogy to the definitions given for crossed modules of Lie algebras (see [7]), we can define the crossed module of *inner biderivations* of $(\mathfrak{n}, \mathfrak{q}, \mu)$, denoted by $\mathrm{InnBider}(\mathfrak{n}, \mathfrak{q}, \mu)$, as $\mathrm{Im}(\varphi, \psi)$, which is obviously an ideal. The crossed module of *outer biderivations*, denoted by $\mathrm{OutBider}(\mathfrak{n}, \mathfrak{q}, \mu)$, is the quotient of $\mathrm{Act}(\mathfrak{n}, \mathfrak{q}, \mu)$ by $\mathrm{InnBider}(\mathfrak{n}, \mathfrak{q}, \mu)$.

Let

$$(0,0,0) \longrightarrow (\mathfrak{n}, \mathfrak{q}, \mu) \longrightarrow (\mathfrak{n}', \mathfrak{q}', \mu') \longrightarrow (\mathfrak{n}'', \mathfrak{q}'', \mu'') \longrightarrow (0,0,0)$$

be a short exact sequence of crossed modules of Leibniz algebras. Then, there exists a homomorphism of Leibniz crossed modules $(\alpha, \beta) \colon (\mathfrak{n}', \mathfrak{q}', \mu') \to$

$\mathrm{Act}(\mathfrak{n},\mathfrak{q},\mu)$ so that the following diagram is commutative:

$$\begin{array}{ccccccccc}
(0,0,0) & \longrightarrow & (\mathfrak{n},\mathfrak{q},\mu) & \longrightarrow & (\mathfrak{n}',\mathfrak{q}',\mu') & \longrightarrow & (\mathfrak{n}'',\mathfrak{q}'',\mu'') & \longrightarrow & (0,0,0) \\
 & & \downarrow & & \downarrow{\scriptstyle (\alpha,\beta)} & & \downarrow & & \\
(0,0,0) & \longrightarrow & \mathrm{InnBider}(\mathfrak{n},\mathfrak{q},\mu) & \longrightarrow & \mathrm{Act}(\mathfrak{n},\mathfrak{q},\mu) & \longrightarrow & \mathrm{OutBider}(\mathfrak{n},\mathfrak{q},\mu) & \longrightarrow & (0,0,0)
\end{array}$$

where (α,β) is defined as $\alpha(n') = (d_{n'}, D_{n'})$ and $\beta(q') = ((\sigma_1^{q'},\theta_1^{q'}),(\sigma_2^{q'},\theta_2^{q'}))$, with

$$d_{n'}(q) = -[q,n'], \qquad D_{n'}(q) = [n',q],$$

and

$$\begin{aligned}
\sigma_1^{q'}(n) &= -[n,q'], & \theta_1^{q'}(n) &= [q',n], \\
\sigma_2^{q'}(q) &= -[q,q'], & \theta_2^{q'}(q) &= [q',q],
\end{aligned}$$

for all $n' \in \mathfrak{n}'$, $q' \in \mathfrak{q}'$, $n \in \mathfrak{n}$, $q \in \mathfrak{q}$.

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Chapter 7

A natural extension of the universal enveloping algebra functor to crossed modules of Leibniz algebras

Abstract

The universal enveloping algebra functor between Leibniz and associative algebras defined by Loday and Pirashvili is extended to crossed modules. We prove that the universal enveloping crossed module of algebras of a crossed module of Leibniz algebras is its natural generalization. Then we construct an isomorphism between the category of representations of a Leibniz crossed module and the category of left modules over its universal enveloping crossed module of algebras. Our approach is particularly interesting since the actor in the category of Leibniz crossed modules does not exist in general, so the technique used in the proof for the Lie case cannot be applied. Finally we move on to the framework of the Loday-Pirashvili category $\mathcal{LM}$ in order to comprehend this universal enveloping crossed module in terms of the Lie crossed modules case.

Reference

R. Fernández-Casado, X. García-Martínez, and M. Ladra, *A natural extension of the universal enveloping algebra functor to crossed modules of Leibniz algebras*, Appl. Categ. Structures, 2016, doi:10.1007/s10485-016-9472-9.

R. Fernández-Casado, X. García-Martínez, and M. Ladra, *Universal enveloping crossed module of Leibniz crossed modules and representations*, Journal of Physics: Conference Series **697** (2016), no. 1, 012007.

7.1 Introduction

Leibniz algebras, which are a non-antisymmetric generalization of Lie algebras, were introduced in 1965 by Bloh [3], who called them D-algebras and referred to the well-known *Leibniz identity* as *differential identity.* In 1993 Loday [19] made them popular and studied their (co)homology. From that moment, many authors have studied this structure, obtaining very relevant algebraic results [21, 22] and applications to Geometry [18, 23] and Physics [12].

Crossed modules of groups were described for the first time by Whitehead in the late 1940s [31] as an algebraic model for path-connected CW-spaces whose homotopy groups are trivial in dimensions greater than 2. From that moment, crossed modules of different algebraic objects, not only groups, have been considered, either as tools or as algebraic structures in their own right. For instance, in [21] crossed modules of Leibniz algebras were defined in order to study cohomology.

Observe that in Ellis's PhD thesis [11] it is proved that, given a category of groups with operations $\mathcal{C}$ such as the categories of associative and Leibniz algebras, crossed modules, cat^1-objects, internal categories and simplicial objects in $\mathcal{C}$ whose Moore complexes are of length 1 are equivalent structures. Note that Ellis's definition of category of groups with operations is more restrictive than the general notion of category of groups with multiple operators introduced by Higgins [14].

Internal categories can be described in terms of what Baez calls strict 2-dimensional objects (see [2] for groups and [1] for Lie algebras). By analogy to Baez's terminology, crossed modules of associative algebras (respectively Leibniz algebras) can be viewed as strict associative 2-algebras [17] (respectively strict Leibniz 2-algebras [13, 29]).

In the case of Lie algebras, the universal enveloping algebra plays two important roles: the category of representations of a Lie algebra is isomorphic to the category of left modules over its universal enveloping algebra and the universal enveloping algebra functor is right adjoint to the Liezation functor. For Leibniz algebras, these roles are played by two different functors: Loday and Pirashvili [21] proved that, given a Leibniz algebra $\mathfrak{p}$, the category of left $\mathtt{UL}(\mathfrak{p})$-modules is isomorphic to the category of $\mathfrak{p}$-representations, where $\mathtt{UL}(\mathfrak{p})$ is the universal enveloping associative algebra of $\mathfrak{p}$. On the other hand, if associative algebras are replaced by dialgebras, there exist a universal enveloping dialgebra functor [20], which is right adjoint to the functor that assigns to every dialgebra its corresponding Leibniz algebra.

Another very interesting point of view on the construction of $\mathtt{UL}(\mathfrak{p})$ is explored in [22]. They introduce the tensor category of linear maps $\mathcal{LM}$, also known as the Loday-Pirashvili category. It is possible to define Lie and associative objects in that category and to construct the universal enveloping algebra. Since a Leibniz algebra can be considered as a Lie object in $\mathcal{LM}$, it is remarkable that $\mathtt{UL}(\mathfrak{p})$ can be obtained via the universal enveloping algebra of the aforementioned Lie object in $\mathcal{LM}$. This approach is especially useful for studying the universal enveloping algebra of a Leibniz algebra in terms of Lie algebras.

As Norrie states in [26], it is surprising the ease of the generalization to crossed modules of many properties satisfied by the objects in the base category. In [8], the universal enveloping dialgebra functor is extended to crossed modules. The aim of this article is to extend to crossed modules the functor $\mathtt{UL}$, to prove that the aforementioned isomorphism between representations of a Leibniz algebra and left modules over its universal enveloping algebra also exists, and to study the 2-dimensional version of $\mathtt{UL}$ in terms of Lie objects in $\mathcal{LM}$.

Observe that the analogous isomorphism between representations and left modules in the case of Lie crossed modules can be easily proved via the actor, but this method cannot be applied in our case, since the actor of a Leibniz crossed module does not necessarily exist [13] (see [6] for the 1-dimensional case). This makes our approach especially interesting.

In Section 7.2 we recall some basic definitions and properties, such as the concept of crossed module of associative and Leibniz algebras, along with the notions of the corresponding cat^1-objects. In Section 7.3 we give proper definitions of left modules over a crossed module of associative algebras and representations of a Leibniz crossed module. In Section 7.4 we describe the generalization to crossed modules of the functor $\mathtt{UL}\colon \mathbf{Lb} \to \mathbf{Alg}$, that is $\mathtt{XUL}\colon \mathbf{XLb} \to \mathbf{XAlg}$, which assigns to every Leibniz crossed module its corresponding universal enveloping crossed module of algebras. Additionally, we prove that $\mathtt{XUL}$ is a natural generalization of $\mathtt{UL}$, in the sense that it commutes (or commutes up to isomorphism) with the two reasonable ways of regarding associative and Leibniz algebras as crossed modules. In Section 7.5 we construct an isomorphism between the categories of representations of a Leibniz crossed module and the left modules over its universal enveloping crossed module of algebras. Finally, in Section 7.6, we introduce Lie and associative crossed modules in the Loday-Pirashvili category and we prove that the factorization of the crossed module $\mathtt{XUL}(\mathfrak{p})$ via Lie crossed modules in $\mathcal{LM}$ also holds in the 2-dimensional case.

Notations and conventions

Throughout the paper, we fix a commutative ring K with unit. All algebras are considered over K. The categories of Lie, Leibniz and (non-unital) associative algebras will be denoted by **Lie**, **Lb** and **Alg**, respectively.

7.2 Preliminaries

Definition 7.2.1 ([19]). A *(right) Leibniz algebra* $\mathfrak{p}$ *over* K is a K-module together with a bilinear operation $[\ ,\ ]\colon \mathfrak{p}\times\mathfrak{p}\to\mathfrak{p}$, called the Leibniz bracket, which satisfies the Leibniz identity:

$$[[p_1,p_2],p_3] = [p_1,[p_2,p_3]] + [[p_1,p_3],p_2],$$

for all $p_1,p_2,p_3\in\mathfrak{p}$. For the opposite structure, that is left Leibniz algebras, the identity is

$$[p_1,[p_2,p_3]] = [[p_1,p_2],p_3] + [p_2,[p_1,p_3]],$$

for all $p_1,p_2,p_3\in\mathfrak{p}$.

A morphism of Leibniz algebras is a K-linear map that preserves the bracket. We will denote by **Lb** the category of Leibniz algebras and morphisms of Leibniz algebras.

If the bracket of a Leibniz algebra $\mathfrak{p}$ happens to be antisymmetric, then $\mathfrak{p}$ is a Lie algebra. Furthermore, every Lie algebra is a Leibniz algebra. For more examples, see [19].

Recall that a Leibniz algebra $\mathfrak{p}$ acts on another Leibniz algebra $\mathfrak{q}$ if there are two bilinear maps $\mathfrak{p}\times\mathfrak{q}\to\mathfrak{q}$, $(p,q)\mapsto[p,q]$ and $\mathfrak{q}\times\mathfrak{p}\to\mathfrak{q}$, $(q,p)\mapsto[q,p]$, satisfying six identities, which are obtained from the Leibniz identity by taking two elements in $\mathfrak{p}$ and one in $\mathfrak{q}$ (three identities) and one element in $\mathfrak{p}$ and two elements in $\mathfrak{q}$ (three identities). See [21] for more details. Given an action of a Leibniz algebra $\mathfrak{p}$ on another Leibniz algebra $\mathfrak{q}$, it is possible to consider the semidirect product $\mathfrak{q} \rtimes \mathfrak{p}$, whose Leibniz structure is given by:

$$[(q_1, p_1), (q_2, p_2)] = ([q_1, q_2] + [p_1, q_2] + [q_1, p_2], [p_1, p_2]),$$

for all (q_1, p_1), $(q_2, p_2) \in \mathfrak{q} \oplus \mathfrak{p}$.

Definition 7.2.2 ([21])**.** A *representation of a Leibniz algebra* $\mathfrak{p}$ is a K-module M equipped with two actions $\mathfrak{p} \times M \to M$, $(p, m) \mapsto [p, m]$ and $M \times \mathfrak{p} \to M$, $(m, p) \mapsto [m, p]$, satisfying the following three axioms:

$$\begin{aligned}
[m, [p_1, p_2]] &= [[m, p_1], p_2] - [[m, p_2], p_1],\\
[p_1, [m, p_2]] &= [[p_1, m], p_2] - [[p_1, p_2], m],\\
[p_1, [p_2, m]] &= [[p_1, p_2], m] - [[p_1, m], p_2],
\end{aligned}$$

for all $m \in M$ and $p_1, p_2 \in \mathfrak{p}$.

A morphism $f\colon M \to N$ of $\mathfrak{p}$-representations is a K-linear map which is compatible with the left and right actions of $\mathfrak{p}$.

Remark 7.2.3. Given a $\mathfrak{p}$-representation M, we can endow the direct sum of K-modules $M \oplus \mathfrak{p}$ with a Leibniz structure such that M is an abelian ideal and $\mathfrak{p}$ is a subalgebra. The converse statement is also true. It is evident that the Leibniz structure of $M \oplus \mathfrak{p}$ is the one of $M \rtimes \mathfrak{p}$, as described previously.

Definition 7.2.4 ([21])**.** Let $\mathfrak{p}^l$ and $\mathfrak{p}^r$ be two copies of a Leibniz algebra $\mathfrak{p}$. We will denote by x_l and x_r the elements of $\mathfrak{p}^l$ and $\mathfrak{p}^r$ corresponding to $x \in \mathfrak{p}$. Consider the tensor K-algebra $\mathrm{T}(\mathfrak{p}^l \oplus \mathfrak{p}^r)$, which is associative and unital. Let I be the two-sided ideal corresponding to the relations:

$$\begin{aligned}
[x, y]_r &= x_r y_r - y_r x_r,\\
[x, y]_l &= x_l y_r - y_r x_l,\\
(y_r + y_l) x_l &= 0.
\end{aligned}$$

for all $x, y \in \mathfrak{p}$. The *universal enveloping algebra of the Leibniz algebra* $\mathfrak{p}$ is the associative and unital algebra

$$\mathtt{UL}(\mathfrak{p}) := \mathrm{T}(\mathfrak{p}^l \oplus \mathfrak{p}^r)/I.$$

This construction defines a functor $\mathtt{UL}\colon \mathbf{Lb} \to \mathbf{Alg}$.

Theorem 7.2.5 ([21])**.** *The category of representations of the Leibniz algebra* $\mathfrak{p}$ *is isomorphic to the category of left modules over* $\mathtt{UL}(\mathfrak{p})$*.*

Proof. Let M be a representation of $\mathfrak{p}$. It is possible to define a left action of $\mathtt{UL}(\mathfrak{p})$ on the K-module M as follows. Given $x_l \in \mathfrak{p}^l$, $x_r \in \mathfrak{p}^r$ and $m \in M$,

$$x_l \cdot m = [x, m], \qquad x_r \cdot m = [m, x].$$

These actions can be extended to an action of $\mathrm{T}(\mathfrak{p}^l \oplus \mathfrak{p}^r)$ by composition and linearity. It is not complicated to check that this way M is equipped with a structure of left $\mathtt{UL}(\mathfrak{p})$-module.

Regarding the converse statement, it is immediate that, starting with a left $\mathtt{UL}(\mathfrak{p})$-module, the restrictions of the actions to $\mathfrak{p}^l$ and $\mathfrak{p}^r$ give two actions of $\mathfrak{p}$ which make M into a representation. □

Although in [21] it is assumed that $\mathfrak{p}$ is free as a K-module, in this theorem the assumption is not necessary.

Recall that a left module over an associative algebra A can be described as a morphism $\alpha\colon A \to \mathrm{End}(M)$, where M is a K-module.

Both **Lb** and **Alg** are categories of interest, notion introduced by Orzech in [27]. See [25] for a proper definition and more examples. Categories of interest are a particular case of categories of groups with operations as described by Ellis [11], for which Porter [28] described the notion of crossed module. Note that the definition of category of groups with operations by Ellis and Porter is a particular case of the general notion of category of groups with multiple operators by Higgins [14]. The following definitions agree with the one given by Porter.

Definition 7.2.6. A *crossed module of Leibniz algebras* (or *Leibniz crossed module*) $(\mathfrak{q}, \mathfrak{p}, \eta)$ is a morphism of Leibniz algebras $\eta\colon \mathfrak{q} \to \mathfrak{p}$ together with an action of $\mathfrak{p}$ on $\mathfrak{q}$ such that

$$\begin{aligned}&\eta([p, q]) = [p, \eta(q)] \qquad \text{and} \qquad \eta([q, p]) = [\eta(q), p],\\ &[\eta(q_1), q_2] = [q_1, q_2] = [q_1, \eta(q_2)],\end{aligned}$$

for all $q, q_1, q_2 \in \mathfrak{q}$, $p \in \mathfrak{p}$.

A *morphism of Leibniz crossed modules* (φ, ψ) from $(\mathfrak{q}, \mathfrak{p}, \eta)$ to $(\mathfrak{q}', \mathfrak{p}', \eta')$ is a pair of Leibniz homomorphisms, $\varphi\colon \mathfrak{q} \to \mathfrak{q}'$ and $\psi\colon \mathfrak{p} \to \mathfrak{p}'$, such that

$$\psi\eta = \eta'\varphi,$$
$$\varphi([p,q]) = [\psi(p), \varphi(q)] \qquad \text{and} \qquad \varphi([q,p]) = [\varphi(q), \psi(p)],$$

for all $q \in \mathfrak{q}$, $p \in \mathfrak{p}$.

Definition 7.2.7. A *crossed module of algebras* (B, A, ρ) is an algebra homomorphism $\rho\colon B \to A$ together with an action of A on B such that

$$\rho(ab) = a\rho(b) \qquad \text{and} \qquad \rho(ba) = \rho(b)a,$$
$$\rho(b_1)b_2 = b_1b_2 = b_1\rho(b_2),$$

for all $a \in A$, $b_1, b_2 \in B$.

A *morphism of crossed modules of algebras* $(\varphi, \psi)\colon (B, A, \rho) \to (B', A', \rho')$ is a pair of algebra homomorphisms, $\varphi\colon B \to B'$ and $\psi\colon A \to A'$, such that

$$\psi\rho = \rho'\varphi,$$
$$\varphi(ba) = \varphi(b)\psi(a) \qquad \text{and} \qquad \varphi(ab) = \psi(a)\varphi(b),$$

for all $b \in B$, $a \in A$.

We will denote by **XLie**, **XLb** and **XAlg** the categories of Lie crossed modules, Leibniz crossed modules and crossed modules of associative algebras, respectively. Crossed modules can be alternatively describe as cat^1-objects, namely:

Definition 7.2.8. A cat^1-*Leibniz algebra* $(\mathfrak{p}_1, \mathfrak{p}_0, s, t)$ consists of a Leibniz algebra $\mathfrak{p}_1$ together with a Leibniz subalgebra $\mathfrak{p}_0$ and structural morphisms $s, t\colon \mathfrak{p}_1 \to \mathfrak{p}_0$ such that

$$s|_{\mathfrak{p}_0} = t|_{\mathfrak{p}_0} = \mathrm{id}_{\mathfrak{p}_0}, \tag{CLb1}$$
$$[\operatorname{Ker} s, \operatorname{Ker} t] = 0 = [\operatorname{Ker} t, \operatorname{Ker} s], \tag{CLb2}$$

Definition 7.2.9. A cat^1-*algebra* (A_1, A_0, σ, τ) consists of an algebra A_1 together with a subalgebra A_0 and structural morphisms $\sigma, \tau\colon A_1 \to A_0$ such that

$$\sigma|_{A_0} = \tau|_{A_0} = \mathrm{id}_{A_0}, \tag{CAs1}$$
$$\operatorname{Ker}\sigma \operatorname{Ker}\tau = 0 = \operatorname{Ker}\tau \operatorname{Ker}\sigma. \tag{CAs2}$$

The notions of cat^1-Leibniz algebra and cat^1-algebra are equivalent to the corresponding crossed modules due to well-known categorical facts. In [15], Janelidze described crossed modules in semi-abelian categories using internal actions in such a way that they are equivalent to internal categories. Note that in this context every internal category is an internal groupoid. On the other hand, cat^1-objects are precisely internal reflexive graphs such that the kernels of the domain and codomain maps commute. Leibniz algebras and associative algebras form semi-abelian categories ([16]), which are therefore Mal'tsev. In [5] it is proved that, for a Mal'tsev category, an internal reflexive graph is an internal category if and only if the kernel equivalence relations of the domain and codomain centralize each other in the sense of Smith ([30]). Moreover they satisfy the so-called "Smith is Huq" condition ([24]), since they are action accessible (see [4] and [25]). Under "Smith is Huq", this condition on the kernel equivalence relations is equivalent to the condition that the kernels of the domain and codomain maps commute. Consequently, in this context cat^1-objects and internal crossed modules are equivalent.

Given a crossed module of Leibniz algebras $(\mathfrak{q}, \mathfrak{p}, \eta)$, the corresponding cat^1-Leibniz algebra is $(\mathfrak{q} \rtimes \mathfrak{p}, \mathfrak{p}, s, t)$, where $s(q, p) = p$ and $t(q, p) = \eta(q) + p$ for all $(q, p) \in \mathfrak{q} \rtimes \mathfrak{p}$. Conversely, given a cat^1-Leibniz algebra $(\mathfrak{p}_1, \mathfrak{p}_0, s, t)$, the corresponding Leibniz crossed module is $t|_{\operatorname{Ker} s} \colon \operatorname{Ker} s \to \mathfrak{p}_0$, with the action of $\mathfrak{p}_0$ on $\operatorname{Ker} s$ induced by the bracket in $\mathfrak{p}_1$. The equivalence for associative algebras is analogous.

The standard functor liezation, $\mathtt{Lie}\colon \mathbf{Lb} \to \mathbf{Lie}$, $\mathfrak{p} \mapsto \mathtt{Lie}(\mathfrak{p})$, where $\mathtt{Lie}(\mathfrak{p})$ is the quotient of $\mathfrak{p}$ by the ideal generated by the elements $[p, p]$, for $p \in \mathfrak{p}$, can be extended to crossed modules $\mathtt{XLie}\colon \mathbf{XLb} \to \mathbf{XLie}$.

Given a Leibniz crossed module $(\mathfrak{q}, \mathfrak{p}, \eta)$ its liezation $\mathtt{XLie}(\mathfrak{q}, \mathfrak{p}, \eta)$ is defined as the crossed module $\left(\frac{\mathtt{Lie}(\mathfrak{q})}{[\mathfrak{q}, \mathfrak{p}]_{\mathtt{x}}}, \mathtt{Lie}(\mathfrak{p}), \overline{\eta}\right)$, where $[\mathfrak{q}, \mathfrak{p}]_{\mathtt{x}}$ is the ideal generated by the elements $[q, p] + [p, q]$, for $p \in \mathfrak{p}, q \in \mathfrak{q}$. This construction can be found in [13].

7.3 Representations of crossed modules

Since our intention is to extend Theorem 7.2.5 to crossed modules, it is necessary to give a proper definition of representations over Leibniz crossed modules and left modules over crossed modules of algebras.

In the case of crossed modules of associative algebras, by analogy to the

1-dimensional situation, left modules can be described via the endomorphism crossed module:

Definition 7.3.1. Let (B, A, ρ) be a crossed module of algebras. A *left (B, A, ρ)-module* is an abelian crossed module of algebras (V, W, δ), that is δ is simply a morphism of K-modules and the action of W on V is trivial, together with a morphism of crossed modules of algebras $(\varphi, \psi)\colon (B, A, \rho) \to (\mathrm{Hom}_K(W, V), \mathrm{End}(V, W, \delta), \Gamma)$.

Note that $\mathrm{End}(V, W, \delta)$ is the algebra of all pairs (α, β), with $\alpha \in \mathrm{End}_K(V)$ and $\beta \in \mathrm{End}_K(W)$, such that $\beta\delta = \delta\alpha$. Furthermore, $\Gamma(d) = (d\delta, \delta d)$ for all $d \in \mathrm{Hom}_K(W, V)$. The action of $\mathrm{End}(V, W, \delta)$ on $\mathrm{Hom}_K(W, V)$ is given by

$$(\alpha, \beta) \cdot d = \alpha d \qquad \text{and} \qquad d \cdot (\alpha, \beta) = d\beta,$$

for all $d \in \mathrm{Hom}_K(W, V)$, $(\alpha, \beta) \in \mathrm{End}(V, W, \delta)$. See [7, 9] for further details.

Let (V, W, δ) and (V', W', δ') be left (B, A, ρ)-modules with the corresponding homomorphisms of crossed modules of algebras $(\varphi, \psi)\colon (B, A, \rho) \to (\mathrm{Hom}_K(W, V), \mathrm{End}(V, W, \delta), \Gamma)$ and $(\varphi', \psi')\colon (B, A, \rho) \to (\mathrm{Hom}_K(W', V'), \mathrm{End}(V', W', \delta'), \Gamma')$. Then a morphism from (V, W, δ) to (V', W', δ') is a pair (f_V, f_W) of morphisms of K-modules $f_V\colon V \to V'$ and $f_W\colon W \to W'$ such that

$$\begin{aligned} f_W\delta &= \delta' f_V, \\ (f_V, f_W)\psi(a) &= \psi'(a)(f_V, f_W), \\ f_V\varphi(b) &= \varphi'(b) f_W, \end{aligned}$$

for all $b \in B$, $a \in A$.

For the categories of crossed modules of groups and Lie algebras, representations can be defined via an object called the actor (see [10, 26]). However this is not the case for Leibniz crossed modules (see [13]). Nevertheless, it is possible to give a definition by equations:

Definition 7.3.2. A *representation of a Leibniz crossed module* $(\mathfrak{q}, \mathfrak{p}, \eta)$ is an abelian Leibniz crossed module (N, M, μ) endowed with:

(i) Actions of the Leibniz algebra $\mathfrak{p}$ (and so $\mathfrak{q}$ via η) on N and M, such that the homomorphism μ is $\mathfrak{p}$-equivariant, that is

$$\mu([p, n]) = [p, \mu(n)], \tag{LbEQ1}$$
$$\mu([n, p]) = [\mu(n), p], \tag{LbEQ2}$$

for all $n \in N$ and $p \in \mathfrak{p}$.

(ii) Two K-bilinear maps $\xi_1 \colon \mathfrak{q} \times M \to N$ and $\xi_2 \colon M \times \mathfrak{q} \to N$ such that

$$\mu\xi_2(m,q) = [m,q], \tag{LbM1a}$$
$$\mu\xi_1(q,m) = [q,m], \tag{LbM1b}$$
$$\xi_2(\mu(n),q) = [n,q], \tag{LbM2a}$$
$$\xi_1\big(q,\mu(n)\big) = [q,n], \tag{LbM2b}$$
$$\xi_2(m,[p,q]) = \xi_2([m,p],q) - [\xi_2(m,q),p], \tag{LbM3a}$$
$$\xi_1([p,q],m) = \xi_2([p,m],q) - [p,\xi_2(m,q)], \tag{LbM3b}$$
$$\xi_2(m,[q,p]) = [\xi_2(m,q),p] - \xi_2([m,p],q), \tag{LbM3c}$$
$$\xi_1([q,p],m) = [\xi_1(q,m),p] - \xi_1(q,[m,p]), \tag{LbM3d}$$
$$\xi_2(m,[q,q']) = [\xi_2(m,q),q'] - [\xi_2(m,q'),q], \tag{LbM4a}$$
$$\xi_1([q,q'],m) = [\xi_1(q,m),q'] - [q,\xi_2(m,q')], \tag{LbM4b}$$
$$\xi_1(q,[p,m]) = -\xi_1(q,[m,p]), \tag{LbM5a}$$
$$[p,\xi_1(q,m)] = -[p,\xi_2(m,q)], \tag{LbM5b}$$

for all $q,q' \in \mathfrak{q}$, $p \in \mathfrak{p}$, $n \in N$ $m,m' \in M$.

A morphism between two representations (N,M,μ) and (N',M',μ') of a Leibniz crossed module $(\mathfrak{q},\mathfrak{p},\eta)$ is a morphism of abelian Leibniz crossed modules $(f_N,f_M) \colon (N,M,\mu) \to (N',M',\mu')$ that preserves the actions together with the morphisms from (ii).

Remark 7.3.3. As in Remark 7.2.3, given a $(\mathfrak{q},\mathfrak{p},\eta)$-representation (N,M,μ), we can obtain a Leibniz crossed module structure on $(N \rtimes \mathfrak{q}, M \rtimes \mathfrak{p}, \mu \oplus \eta)$ where (N,M,μ) is an abelian crossed ideal and $(\mathfrak{q},\mathfrak{p},\eta)$ is a crossed submodule of $(N \rtimes \mathfrak{q}, M \rtimes \mathfrak{p}, \mu \oplus \eta)$ respectively. The converse statement is also true. Moreover, a representation can be seen as an action of $(\mathfrak{q},\mathfrak{p},\eta)$ over an abelian Leibniz crossed module (N,M,μ) in the sense of [13].

7.4 Universal enveloping crossed module of algebras of a Leibniz crossed module

Let $(\mathfrak{q},\mathfrak{p},\eta)$ be a Leibniz crossed module and consider its corresponding cat^1-Leibniz algebra

$$\mathfrak{q} \rtimes \mathfrak{p} \overset{s}{\underset{t}{\rightrightarrows}} \mathfrak{p}\ ,$$

with $s(q,p) = p$ and $t(q,p) = \eta(q) + p$ for all $(q,p) \in \mathfrak{q} \rtimes \mathfrak{p}$. Now, if we apply $\mathtt{UL}$ to the previous diagram, we get

$$\mathtt{UL}(\mathfrak{q} \rtimes \mathfrak{p}) \underset{\mathtt{UL}(t)}{\overset{\mathtt{UL}(s)}{\rightrightarrows}} \mathtt{UL}(\mathfrak{p}) \ .$$

Although it is true that $\mathtt{UL}(s)|_{\mathtt{UL}(\mathfrak{p})} = \mathtt{UL}(t)|_{\mathtt{UL}(\mathfrak{p})} = \mathrm{id}_{\mathtt{UL}(\mathfrak{p})}$, in general, the second condition for cat^1-algebras (CAs2) is not satisfied. Nevertheless, we can consider the quotient $\overline{\mathtt{UL}}(\mathfrak{q} \rtimes \mathfrak{p}) = \mathtt{UL}(\mathfrak{q} \rtimes \mathfrak{p})/\mathcal{X}$, where $\mathcal{X} = \operatorname{Ker} \mathtt{UL}(s) \operatorname{Ker} \mathtt{UL}(t) + \operatorname{Ker} \mathtt{UL}(t) \operatorname{Ker} \mathtt{UL}(s)$, and the induced morphisms $\overline{\mathtt{UL}}(s)$ and $\overline{\mathtt{UL}}(t)$. In this way, the diagram

$$\overline{\mathtt{UL}}(\mathfrak{q} \rtimes \mathfrak{p}) \underset{\overline{\mathtt{UL}}(t)}{\overset{\overline{\mathtt{UL}}(s)}{\rightrightarrows}} \mathtt{UL}(\mathfrak{p})$$

is clearly a cat^1-algebra. This construction is just a particular case of the general categorical way of obtaining an internal groupoid from a reflexive graph in a semi-abelian category satisfying the “Smith is Huq” condition.

Note that $\mathtt{UL}(\mathfrak{p})$ can be regarded as a subalgebra of $\overline{\mathtt{UL}}(\mathfrak{q} \rtimes \mathfrak{p})$.

We can now define $\mathtt{XUL}(\mathfrak{q}, \mathfrak{p}, \eta)$ as the crossed module of associative algebras given by $(\operatorname{Ker} \overline{\mathtt{UL}}(s), \mathtt{UL}(\mathfrak{p}), \overline{\mathtt{UL}}(t)|_{\operatorname{Ker} \overline{\mathtt{UL}}(s)})$. This construction defines a functor $\mathtt{XUL}\colon$ **XLb** → **XAlg**.

Immediately below we prove a very helpful lemma which gives us a proper description of $\operatorname{Ker} \mathtt{UL}(s)$.

Lemma 7.4.1. *The elements of the form* $(q_1, p_1) \otimes \cdots \otimes (q_k, p_k)$ *such that there exists* $1 \leqslant i \leqslant k$ *with* $p_i = 0$*, generate* $\operatorname{Ker} \mathtt{UL}(s)$*.*

Proof. Let J be the ideal of $\mathrm{T}(\mathfrak{p}^l \oplus \mathfrak{p}^r)$ generated by the three relations of Definition 7.2.4. Let I be the ideal of $\mathrm{T}\left((\mathfrak{q} \rtimes \mathfrak{p})^l \oplus (\mathfrak{q} \rtimes \mathfrak{p})^r\right)$ generated by the preimage by $\mathrm{T}(s \oplus s)$ of those relations. Then I is the ideal generated by

$$\begin{aligned}
&(q_1, p)_r \otimes (q_2, p')_r - (q_3, p')_r \otimes (q_4, p)_r - (q_5, [p, p'])_r, \\
&(q_1, p)_r \otimes (q_2, p')_l - (q_3, p')_l \otimes (q_4, p)_r - (q_5, [p, p'])_l, \\
&(q_1, p)_r \otimes (q_2, p')_l + (q_1, p)_l \otimes (q_2, p')_l.
\end{aligned}$$

Additionally, the kernel of $\mathrm{T}(s \oplus s)$ is generated by elements as those in the statement of this lemma and by elements of the form $(q_1, p_1)_{\alpha_1} \otimes \cdots \otimes (q_k, p_k)_{\alpha_k} - (q'_1, p_1)_{\alpha_1} \otimes \cdots \otimes (q'_k, p_k)_{\alpha_k}$, where α_k can be r or l. Since $\mathrm{T}(s \oplus s)$

is surjective, the kernel of $\mathtt{UL}(s)$ will be generated by I and $\operatorname{Ker}\mathrm{T}(s \oplus s)$. Let us check that all these generators are of the claimed form.

Given $(q_1, p')_{\alpha_1} \otimes (q_2, p)_{\alpha_2} - (q_3, p')_{\alpha_1} \otimes (q_4, p)_{\alpha_2} \in \operatorname{Ker}\mathrm{T}(s \oplus s)$, we have that

$$\begin{aligned}(q_1, p')_{\alpha_1} &\otimes (q_2, p)_{\alpha_2} - (q_3, p')_{\alpha_1} \otimes (q_4, p)_{\alpha_2} \\ &= (q_1, p')_{\alpha_1} \otimes (q_2, p)_{\alpha_2} - (q_3, p')_{\alpha_1} \otimes (q_4, p)_{\alpha_2} \\ &\quad + (q_1, p')_{\alpha_1} \otimes (q_4, p)_{\alpha_2} - (q_1, p')_{\alpha_1} \otimes (q_4, p)_{\alpha_2} \\ &= (q_1, p')_{\alpha_1} \otimes (q_2 - q_4, 0)_{\alpha_2} + (q_1 - q_3, 0)_{\alpha_1} \otimes (q_4, p)_{\alpha_2}.\end{aligned}$$

By induction one can easily derive that the elements in $\operatorname{Ker}\mathrm{T}(s \oplus s)$ are of the expected form.

Let us take $(q_1, p)_r \otimes (q_2, p')_r - (q_3, p')_r \otimes (q_4, p)_r - (q_5, [p, p'])_r \in I$. Then,

$$\begin{aligned}(q_1,p)_r &\otimes (q_2, p')_r - (q_3, p')_r \otimes (q_4, p)_r - (q_5, [p, p'])_r \\ &= (q_1, p)_r \otimes (q_2, p')_r - (q_4, p)_r \otimes (q_3, p')_r + [(q_4, p), (q_3, p')]_r - (q_5, [p, p'])_r \\ &= (q_1, p)_r \otimes (q_2, p')_r - (q_4, p)_r \otimes (q_3, p')_r + (q', [p, p'])_r - (q_5, [p, p'])_r \\ &= (q_1, p)_r \otimes (q_2, p')_r - (q_4, p)_r \otimes (q_3, p')_r + (q' - q_5, 0)_r,\end{aligned}$$

and then we proceed as in the previous case. For the second and third identities the argument is similar. □

Observe that there are full embeddings

$$\mathtt{I}_0, \mathtt{I}_1 \colon \mathsf{Alg} \longrightarrow \mathsf{XAlg} \qquad (\text{resp.} \quad \mathtt{J}_0, \mathtt{J}_1 \colon \mathsf{Lb} \longrightarrow \mathsf{XLb})$$

defined, for an associative algebra A (resp. for a Leibniz algebra $\mathfrak{p}$), by $\mathtt{I}_0(A) = (\{0\}, A, 0)$, $\mathtt{I}_1(A) = (A, A, \mathrm{id}_A)$ $\big($resp. $\mathtt{J}_0(\mathfrak{p}) = (\{0\}, \mathfrak{p}, 0)$, $\mathtt{J}_1(\mathfrak{p}) = (\mathfrak{p}, \mathfrak{p}, \mathrm{id}_{\mathfrak{p}})\big)$, where the action is given by multiplication in A (resp. $\mathfrak{p}$). The functor $\mathtt{XUL}\colon \mathsf{XLb} \to \mathsf{XAlg}$ is a natural generalization of the functor $\mathtt{UL}$, in the sense that it makes the following diagram commute,

$$\begin{array}{ccc} \mathsf{Lb} & \xrightarrow{\mathtt{J}_0} & \mathsf{XLb} \\ {\scriptstyle \mathtt{UL}}\downarrow & & \downarrow{\scriptstyle \mathtt{XUL}} \\ \mathsf{Alg} & \xrightarrow[\mathtt{I}_0]{} & \mathsf{XAlg} \end{array}$$

Regarding the embeddings $\mathtt{I}_1$ and $\mathtt{J}_1$, we have the following result.

Proposition 7.4.2. *There is a natural isomorphism of functors*

$$\mathit{XUL} \circ J_1 \cong I_1 \circ \mathit{UL}\,.$$

Proof. Let $\mathfrak{p} \in \mathbf{Lb}$. It is necessary to prove that $\mathtt{XUL}(\mathfrak{p}, \mathfrak{p}, \mathrm{id}_{\mathfrak{p}})$ is naturally isomorphic to $(\mathtt{UL}(\mathfrak{p}), \mathtt{UL}(\mathfrak{p}), \mathrm{id}_{\mathtt{UL}(\mathfrak{p})})$. In order to do so, we will show that $(\overline{\mathtt{UL}}(t)|_{\operatorname{Ker}\overline{\mathtt{UL}}(s)}, \mathrm{id}_{\mathtt{UL}(\mathfrak{p})})$ is an isomorphism of crossed modules of algebras between $(\operatorname{Ker}\overline{\mathtt{UL}}(s), \mathtt{UL}(\mathfrak{p}), \overline{\mathtt{UL}}(t)|_{\operatorname{Ker}\overline{\mathtt{UL}}(s)})$ and $(\mathtt{UL}(\mathfrak{p}), \mathtt{UL}(\mathfrak{p}), \mathrm{id}_{\mathtt{UL}(\mathfrak{p})})$.

It is easy to check that $(\overline{\mathtt{UL}}(t)|_{\operatorname{Ker}\overline{\mathtt{UL}}(s)}, \mathrm{id}_{\mathtt{UL}(\mathfrak{p})})$ is indeed a morphism of crossed modules of algebras. Recall that the first step in the construction of $\mathtt{XUL}(\mathfrak{p}, \mathfrak{p}, \mathrm{id}_{\mathfrak{p}})$ requires us to consider the cat^1-Leibniz algebra

$$\mathfrak{p} \rtimes \mathfrak{p} \underset{t}{\overset{s}{\rightrightarrows}} \mathfrak{p}\ ,$$

with $s(p, p') = p'$ and $t(p, p') = p + p'$ for all $p, p' \in \mathfrak{p}$. Let us define the Leibniz homomorphism $\epsilon\colon \mathfrak{p} \to \mathfrak{p} \rtimes \mathfrak{p}$, $\epsilon(p) = (p, 0)$. It is clear that $s\epsilon = 0$ and $t\epsilon = \mathrm{id}_{\mathfrak{p}}$.

The next step is to apply the functor $\mathtt{UL}$ on the previous cat^1-Leibniz algebra and take the quotient of $\mathtt{UL}(\mathfrak{p} \rtimes \mathfrak{p})$ by $\mathcal{X} = \operatorname{Ker}\mathtt{UL}(s)\operatorname{Ker}\mathtt{UL}(t) + \operatorname{Ker}\mathtt{UL}(t)\operatorname{Ker}\mathtt{UL}(s)$ in order to guarantee that we have a cat^1-algebra. In the next diagram of algebras,

$$\begin{array}{ccccc}
\mathtt{UL}(\mathfrak{p}) & \xrightarrow{\ \mathtt{UL}(\epsilon)\ } & \mathtt{UL}(\mathfrak{p} \rtimes \mathfrak{p}) & \underset{\mathtt{UL}(t)}{\overset{\mathtt{UL}(s)}{\rightrightarrows}} & \mathtt{UL}(\mathfrak{p}) \\
 & & \downarrow{\scriptstyle \pi} & \nearrow\!\!\nearrow{\scriptstyle \overline{\mathtt{UL}}(s),\ \overline{\mathtt{UL}}(t)} & \\
 & & \overline{\mathtt{UL}}(\mathfrak{p} \rtimes \mathfrak{p}) & &
\end{array}$$

where π is the canonical projection, it is easy to see that $\overline{\mathtt{UL}}(s)\pi\,\mathtt{UL}(\epsilon) = \mathtt{UL}(s)\,\mathtt{UL}(\epsilon) = \mathtt{UL}(s\epsilon) = 0$ and $\overline{\mathtt{UL}}(t)\pi\,\mathtt{UL}(\epsilon) = \mathtt{UL}(t)\,\mathtt{UL}(\epsilon) = \mathtt{UL}(t\epsilon) = \mathrm{id}_{\mathtt{UL}(\mathfrak{p})}$. Hence $\pi\,\mathtt{UL}(\epsilon)$ takes values in $\operatorname{Ker}\overline{\mathtt{UL}}(s)$ and it is a right inverse for $\overline{\mathtt{UL}}(t)|_{\operatorname{Ker}\overline{\mathtt{UL}}(s)}$.

Now we need to show that $\pi\,\mathtt{UL}(\epsilon)\overline{\mathtt{UL}}(t) = \mathrm{id}_{\operatorname{Ker}\overline{\mathtt{UL}}(s)}$. Note that $\mathcal{X} \subset \operatorname{Ker}\mathtt{UL}(s)$, so $\operatorname{Ker}\overline{\mathtt{UL}}(s) = \operatorname{Ker}\mathtt{UL}(s)/\mathcal{X}$ and, as proved in Lemma 7.4.1, $\operatorname{Ker}\mathtt{UL}(s)$ is generated by all the elements of the form

$$(p_1, p_1') \otimes \cdots \otimes (p_i, 0) \otimes \cdots \otimes (p_k, p_k') \tag{7.4.1}$$

with $p_j, p_j' \in \mathfrak{p}$, $1 \leqslant i, j \leqslant k$. By the definition of $\mathtt{UL}(t)$ and $\mathtt{UL}(\epsilon)$, the value of $\mathtt{UL}(\epsilon)\,\mathtt{UL}(t)$ on (7.4.1) is

$$(p_1 + p_1', 0) \otimes \cdots \otimes (p_i, 0) \otimes \cdots \otimes (p_k + p_k', 0). \tag{7.4.2}$$

Furthermore, one can easily derive that, in $\operatorname{Ker}\mathtt{UL}(s)/\mathcal{X}$,

$$\begin{aligned}&(p_1+p_1',0)\otimes\cdots\otimes(p_i,0)\otimes\cdots\otimes(p_k+p_k',0)\\&=(p_1,p_1')\otimes\cdots\otimes(p_i,0)\otimes\cdots\otimes(p_k+p_k',0),\end{aligned}$$

By applying the same procedure as many times as required, one can deduce that

$$\begin{aligned}&(p_1+p_1',0)\otimes\cdots\otimes(p_i,0)\otimes\cdots\otimes(p_k+p_k',0)\\&=(p_1,p_1')\otimes\cdots\otimes(p_i,0)\otimes\cdots\otimes(p_k,p_k').\end{aligned}$$

Thus, the elements (7.4.1) and (7.4.2) are equal in $\operatorname{Ker}\mathtt{UL}(s)/\mathcal{X}$ and it follows that

$$\pi\,\mathtt{UL}(\epsilon)\overline{\mathtt{UL}}(t)|_{\operatorname{Ker}\overline{\mathtt{UL}}(s)}=\mathrm{id}_{\operatorname{Ker}\overline{\mathtt{UL}}(s)}\,.$$

Therefore we have found an inverse for the morphism of crossed modules of algebras $(\overline{\mathtt{UL}}(t)|_{\operatorname{Ker}\overline{\mathtt{UL}}(s)},\mathrm{id}_{\mathtt{UL}(\mathfrak{p})})$. It is fairly easy to prove that this construction is natural. □

7.5 Isomorphism between the categories of representations

In this section, we give the construction of an isomorphism between the categories of representations of a Leibniz crossed module and left modules over its corresponding universal enveloping crossed module of algebras. Recall that the method used in the proof of the equivalent result in the case of Lie algebras cannot be applied in our case due to the lack of actor in the category of Leibniz crossed modules.

Theorem 7.5.1. *The category of representations of a Leibniz crossed module* $(\mathfrak{q},\mathfrak{p},\eta)$ *is isomorphic to the category of left modules over its universal enveloping crossed module of algebras* $\mathtt{XUL}(\mathfrak{q},\mathfrak{p},\eta)$.

Proof. Let (N,M,μ) be a left $(\operatorname{Ker}\overline{\mathtt{UL}}(s),\mathtt{UL}(\mathfrak{p}),\overline{\mathtt{UL}}(t)|_{\operatorname{Ker}\overline{\mathtt{UL}}(s)})$-module. Then we have a homomorphism

$$(\varphi,\psi)\colon(\operatorname{Ker}\overline{\mathtt{UL}}(s),\mathtt{UL}(\mathfrak{p}),\overline{\mathtt{UL}}(t)|_{\operatorname{Ker}\overline{\mathtt{UL}}(s)})\to(\operatorname{Hom}_K(M,N),\operatorname{End}(N,M,\mu),\Gamma),$$

such that $\psi\circ\overline{\mathtt{UL}}(t)|_{\operatorname{Ker}\overline{\mathtt{UL}}(s)}=\Gamma\circ\varphi$, $\varphi(ba)=\varphi(b)\psi(a)$ and $\varphi(ab)=\psi(a)\varphi(b)$. We need to define actions of $\mathfrak{p}$ on N and M satisfying (LbEQ1) and (LbEQ2) and we need to define $\xi_1 \colon \mathfrak{q} \times M \to N$ and $\xi_2 \colon M \times \mathfrak{q} \to N$ satisfying identities (LbM1a)–(LbM5b).

We define the actions of $\mathfrak{p}$ on N and M as those induced by $\psi \colon \mathtt{UL}(\mathfrak{p}) \to \mathrm{End}(N, M, \mu)$ as in Theorem 7.2.5. The identities (LbEQ1) and (LbEQ2) follow from the properties of $\mathrm{End}(N, M, \mu)$. We define the morphisms ξ_1 and ξ_2 by $\xi_1(q, m) = \varphi\big((q, 0)_l\big)(m)$ and $\xi_2(m, q) = \varphi\big((q, 0)_r\big)(m)$. The identities (LbM1a), (LbM2a) and (LbM1b), (LbM2b) follow from the commutative square $\psi \circ \overline{\mathtt{UL}}(t)|_{\mathrm{Ker}\,\overline{\mathtt{UL}}(s)} = \Gamma \circ \varphi$ applied to the elements $(q, 0)_r$ and $(q, 0)_l$ respectively. Given the element $([p, q], 0)_r \in \mathrm{Ker}\,\overline{\mathtt{UL}}(s)$, we have that

$$([p, q], 0)_r = [(0, p)_r, (q, 0)_r] = (0, p)_r(q, 0)_r - (q, 0)_r(0, p)_r.$$

Applying φ to this relation and using that $\varphi(ba) = \varphi(b)\psi(a)$ and $\varphi(ab) = \psi(a)\varphi(b)$ we obtain $\varphi\big(([p, q], 0)_r\big) = \psi_2(p_r)\varphi\big((q, 0)_r\big) - \varphi\big((q, 0)_r\big)\psi_1(p_r)$ which implies (LbM3a). Proceeding in the same way for the elements $([p, q], 0)_l$, $([q, p], 0)_r$ and $([q, p], 0)_l$ we check that identities (LbM3b), (LbM3c) and (LbM3d) are satisfied. Doing a similar argument on the elements $([q, q'], 0)_r$ and $([q, q'], 0)_l$ we obtain identities (LbM4a) and (LbM4b). Applying φ to the relations

$$(0, p)_l(q, 0)_l = -(0, p)_r(q, 0)_l \qquad \text{and} \qquad (q, 0)_l(0, p)_l = -(q, 0)_r(0, p)_l$$

we have identities (LbM5a) and (LbM5b) respectively.

Conversely, let (N, M, μ) be a $(\mathfrak{q}, \mathfrak{p}, \eta)$-representation. We need to construct a morphism of crossed modules of algebras (φ, ψ) from $\mathtt{XUL}(\mathfrak{q}, \mathfrak{p}, \eta)$ to $(\mathrm{Hom}_K(M, N), \mathrm{End}(N, M, \mu), \Gamma)$. The homomorphism $\psi = (\psi_1, \psi_2) \colon \mathtt{UL}(\mathfrak{p}) \to \mathrm{End}(N, M, \mu)$ is the homomorphism induced by the actions of $\mathfrak{p}$ on N and M as in Theorem 7.2.5. It is well defined due to identities (LbEQ1) and (LbEQ2). Consider the homomorphism of K-modules $\Phi \colon (\mathfrak{q} \rtimes \mathfrak{p})^l \oplus (\mathfrak{q} \rtimes \mathfrak{p})^r \to \mathrm{Hom}_K(N \oplus M, N \oplus M)$ defined by

$$\Phi(q, p)_l(n, m) = ([q, n] + [p, n] + \xi_1(q, m), [p, m]),$$
$$\Phi(q, p)_r(n, m) = ([n, q] + [n, p] + \xi_2(m, q), [m, p]).$$

Note that they can also be rewritten as

$$\Phi(q, p)_l(n, m) = \big(t(q, p)_l(n) + \xi_1(q, m), s(q, p)_l(m)\big),$$
$$\Phi(q, p)_r(n, m) = \big(t(q, p)_r(n) + \xi_2(m, q), s(q, p)_r(m)\big).$$

By the universal property of the tensor algebra, there is a unique homomorphism

$$\mathrm{T}(\Phi)\colon\ \mathrm{T}\left((\mathfrak{q}\rtimes\mathfrak{p})^l\oplus(\mathfrak{q}\rtimes\mathfrak{p})^r\right)\to\mathrm{Hom}_K(N\oplus M,N\oplus M),$$

commuting with the inclusion.

We consider the projection $\pi\colon\ \mathrm{Hom}_K(N\oplus M,N\oplus M)\to\mathrm{Hom}_K(M,N\oplus M)$, where $\pi(f)(m)=f(0,m)$, and denote $\varphi'=\pi\circ\mathrm{T}(\Phi)$. Given an element of the form $(q',p')_r(q,p)_r-(q,p)_r(q',p')_r+[(q,p)_r,(q',p')_r]$ we obtain that

$$\begin{aligned}\varphi'\big((q',p')_r(q,p)_r-(q,p)_r(q',p')_r+[(q,p)_r,(q',p')_r]\big)(m)\\ =[\xi_2(m,q'),q]+\big[[m,p'],q\big]+[\xi_2(m,q'),p]\\ -[\xi_2(m,q),q']-\big[[m,p],q'\big]-[\xi_2(m,q),p']\\ +\xi_2(m,[q,q'])+\xi_2(m,[q,p'])+\xi_2(m,[q',p])=0,\end{aligned}$$

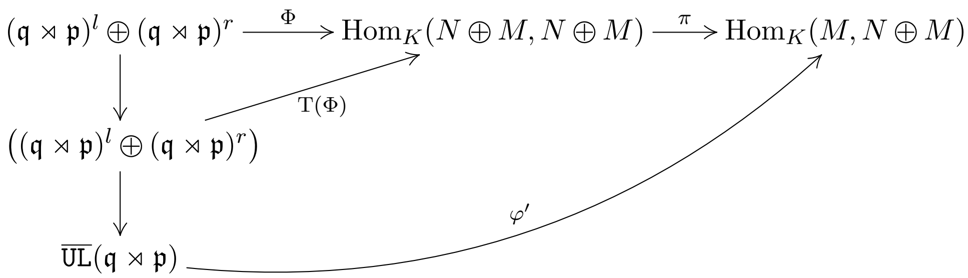
by the properties of ξ_2.

Analogously, it is possible to prove that φ' vanishes on the other two relations of the universal enveloping algebra. Then φ' factors through $\mathtt{UL}(\mathfrak{q}\rtimes\mathfrak{p})$. In order to ease notation we will refer to it as φ' as well.

By definition it is clear that $\varphi'|_{\mathrm{Ker}\,\mathtt{UL}(s)}(M)\subseteq N$ and $\mathrm{T}(\Phi)\big((\mathfrak{q}\rtimes\mathfrak{p})^l\oplus(\mathfrak{q}\rtimes\mathfrak{p})^r\big)(N)=\mathrm{T}(t)\big((\mathfrak{q}\rtimes\mathfrak{p})^l\oplus(\mathfrak{q}\rtimes\mathfrak{p})^r\big)(N)$. Then φ' factors through $\mathrm{Ker}\,\mathtt{UL}(t)\,\mathrm{Ker}\,\mathtt{UL}(s)$. Moreover, we have that

$$\begin{aligned}\varphi'\big((q,p)_r(q',p')_r\big)(m)&=\big([\xi_2(m,q),q']+[\xi_2(m,q),p']+\xi_2([m,p],q),\big[[m,p],p'\big]\big)\\ &=\big(\xi_2(m,[t(q,p)_r,q'])+[\xi_2(m,q'),t(q,p)_r]\\ &\quad+[\xi_2(m,q),s(q',p')_r],s(q',p')_rs(q,p)_rm\big).\end{aligned}$$

Extending this argument we see that φ' also factors through $\mathrm{Ker}\,\mathtt{UL}(s)\,\mathrm{Ker}\,\mathtt{UL}(t)$.



Therefore, φ will be the restriction of φ' to $\operatorname{Ker}\overline{\mathtt{UL}}(\mathfrak{q} \rtimes \mathfrak{p})$ and it will take values in $\operatorname{Hom}_K(M, N)$, that is

$$\varphi\colon \operatorname{Ker}\overline{\mathtt{UL}}(\mathfrak{q} \rtimes \mathfrak{p}) \to \operatorname{Hom}_K(M, N).$$

With these definitions of φ and ψ, to check that

$$(\varphi, \psi)\colon (\operatorname{Ker}\overline{\mathtt{UL}}(s), \mathtt{UL}(\mathfrak{p}), \overline{\mathtt{UL}}(t)|_{\operatorname{Ker}\overline{\mathtt{UL}}(s)}) \to (\operatorname{Hom}_K(M, N), \operatorname{End}(N, M, \mu), \Gamma)$$

is a morphism of crossed modules of algebras is now a matter of straightforward computations. □

7.6 Relation with the Loday-Pirashvili category

In [22], Loday and Pirashvili introduced a very interesting way to see Leibniz algebras as Lie algebras over another tensor category different from K-Mod, the tensor category of linear maps, denoted by $\mathcal{LM}$. In this section we will extend this construction to crossed modules and check that the relation with the universal enveloping algebra still holds in the 2-dimensional case.

Definition 7.6.1 ([22])**.** Let M and $\mathfrak{g}$ be K-modules. The objects in $\mathcal{LM}$ are K-module homomorphisms $(M \xrightarrow{\alpha} \mathfrak{g})$. In order to ease notation we will simply write $(M, \mathfrak{g})$ if there is no possible confusion. Given two objects $M \xrightarrow{\alpha} \mathfrak{g}$ and $N \xrightarrow{\beta} \mathfrak{h}$, an arrow is a pair of K-module homomorphisms $\varrho_1\colon M \to N$ and $\varrho_2\colon \mathfrak{g} \to \mathfrak{h}$ such that $\beta \circ \varrho_1 = \varrho_2 \circ \alpha$. $\mathcal{LM}$ is a tensor category with the tensor product defined as

$$(M \xrightarrow{\alpha} \mathfrak{g}) \otimes (N \xrightarrow{\beta} \mathfrak{h}) = \big((M \otimes \mathfrak{h}) \oplus (\mathfrak{g} \otimes N) \xrightarrow{\alpha\otimes 1_{\mathfrak{h}} + 1_{\mathfrak{g}}\otimes\beta} \mathfrak{g} \otimes \mathfrak{h}\big).$$

An *associative algebra* in $\mathcal{LM}$ is an object $(A \xrightarrow{\beta} R)$ where R is an associative K-algebra, A is a R-bimodule and β is a homomorphism of R-bimodules.

A *Lie algebra* in $\mathcal{LM}$ is an object $(M \xrightarrow{\alpha} \mathfrak{g})$ where $\mathfrak{g}$ is a Lie algebra, M is a right $\mathfrak{g}$-representation and α is $\mathfrak{g}$-equivariant. Given a Lie algebra object in $\mathcal{LM}$, its *universal enveloping algebra* in $\mathcal{LM}$ is $(\mathtt{U}(\mathfrak{g}) \otimes M \to \mathtt{U}(\mathfrak{g}))$, $1 \otimes m \mapsto \alpha(m)$. The action is given by

$$g(x \otimes m) = gx \otimes m \qquad \text{and} \qquad (x \otimes m)g = xg \otimes m + x \otimes [m, g],\ g, x \in \mathfrak{g},\ m \in M.$$

A Leibniz algebra $\mathfrak{p}$ can be viewed as a Lie algebra object in $\mathcal{LM}$, namely $\mathfrak{p} \to \mathtt{Lie}(\mathfrak{p})$.

Definition 7.6.2. Let $(A \xrightarrow{\alpha} R)$ and $(B \xrightarrow{\beta} S)$ be two associative algebras in $\mathcal{LM}$. We say there is an *action* of (A, R) on (B, S) if we have the following:

- An action of algebras of R on S;
- an R-bimodule structure on B, compatible with the action of S;
- two homomorphisms $\xi_1 \colon A \otimes_R S \to B$ and $\xi_2 \colon S \otimes_R A \to B$, such that $\xi_1(a, s)s' = \xi_1(a, ss')$, $s\xi_1(a, s') = \xi_2(s, a)s'$, $s\xi_2(s', a) = \xi_2(ss', a)$;
- β is also a homomorphism of R-bimodules, such that $\beta\big(\xi_1(a, s)\big) = \alpha(a)s$ and $\beta\big(\xi_2(s, a)\big) = s\alpha(a)$.

A *crossed module of associative algebras* in $\mathcal{LM}$ is an arrow $(\omega_1, \omega_2) \colon (B, S) \to (A, R)$ and an action of (A, R) on (B, S) such that

- ω_2 with the action of R on S is a crossed module of associative algebras;
- ω_1 is a homomorphism of R-bimodules satisfying $a\omega_2(s) = \omega_1\big(\xi_1(a, s)\big)$, $\omega_2(s)a = \omega_1\big(\xi_2(s, a)\big)$, $\xi_1\big(\omega_1(b), s\big) = b\omega_2(s) = bs$ and $\xi_2\big(s, \omega_1(b)\big) = \omega_2(s)b = sb$.

Let $(\omega_1, \omega_2) \colon (B, S, \beta) \to (A, R, \alpha)$ be a crossed module of algebras in $\mathcal{LM}$. We can associate to it the crossed module of algebras

$$(B \oplus S, A \oplus R, \omega_1 \oplus \omega_2), \tag{7.6.1}$$

where

$$\begin{aligned} (a, r)(a', r') &= (\alpha(a)a' + ar' + ra', rr'), \qquad && a, a' \in A, \ r, r' \in R, \\ (b, s)(b', s') &= (\beta(b)b' + bs' + sb', ss'), \qquad && b, b' \in B, \ s, s' \in S. \end{aligned}$$

Definition 7.6.3. Let $(M \xrightarrow{\alpha} \mathfrak{g})$ and $(N \xrightarrow{\beta} \mathfrak{h})$ be two Lie algebras in $\mathcal{LM}$. We say there is a *right action* of $(M, \mathfrak{g})$ on $(N, \mathfrak{h})$ if we have the following:

- Compatible right Lie actions of $\mathfrak{g}$ on $\mathfrak{h}$ and N;
- a homomorphism $\xi \colon M \otimes \mathfrak{h} \to N$ such that $[\xi(m, h), g] = \xi([m, g], h) + \xi(m, [h, g])$ and $\xi(m, [h, h']) = [\xi(m, h), h'] - [\xi(m, h'), h]$;

- β is $\mathfrak{g}$-equivariant and it satisfies that $\beta\big(\xi(m,h)\big) = [\alpha(m), h]$.

A *crossed module of Lie algebras* in $\mathcal{LM}$ is an arrow $(\varrho_1, \varrho_2)\colon (N, \mathfrak{h}) \to (M, \mathfrak{g})$ and an action of $(M, \mathfrak{g})$ on $(N, \mathfrak{h})$ such that

- ϱ_2 with the right Lie action of $\mathfrak{g}$ on $\mathfrak{h}$ is a crossed module of Lie algebras;
- ϱ_1 is a $\mathfrak{g}$-equivariant homomorphism such that $\varrho_1\big(\xi(m,h)\big) = [m, \varrho_2(h)]$ and $[n,h] = \xi(\varrho_1(n), h) = [n, \varrho_2(h)]$.

Let $(\varrho_1, \varrho_2)\colon (N, \mathfrak{h}) \to (M, \mathfrak{g})$ be a Lie crossed module in $\mathcal{LM}$. We construct the semidirect product in $\mathcal{LM}$ by obtaining a Lie object $(N \oplus M, \mathfrak{h} \rtimes \mathfrak{g})$, where $[(n,m),(h,g)] = ([n,h]+[n,g]+\xi(m,h), [m,g])$. Let (s_1, s_2) and (t_1, t_2) be two arrows in $\mathcal{LM}$

$$\begin{array}{ccc} N \oplus M & \overset{s_1}{\underset{t_1}{\rightrightarrows}} & M \\ {\scriptstyle \beta\oplus\alpha}\downarrow & & \downarrow{\scriptstyle \alpha} \\ \mathfrak{h} \rtimes \mathfrak{g} & \overset{s_2}{\underset{t_2}{\rightrightarrows}} & \mathfrak{g} \end{array}$$

where $s_1(n,m) = m$, $s_2(h,g) = g$ and $t_1(n,m) = \varrho_1(n) + m$, $t_2(h,g) = \varrho_2(h) + g$. We apply the universal enveloping algebra functor in $\mathcal{LM}$ to the previous diagram.

$$\begin{array}{ccc} \mathtt{U}(\mathfrak{h} \rtimes \mathfrak{g}) \otimes (N \oplus M) & \overset{\mathtt{U}(s_1)}{\underset{\mathtt{U}(t_1)}{\rightrightarrows}} & \mathtt{U}(\mathfrak{g}) \otimes M \\ {\scriptstyle \mathtt{U}(\beta\oplus\alpha)}\downarrow & & \downarrow{\scriptstyle \mathtt{U}(\alpha)} \\ \mathtt{U}(\mathfrak{h} \rtimes \mathfrak{g}) & \overset{\mathtt{U}(s_2)}{\underset{\mathtt{U}(t_2)}{\rightrightarrows}} & \mathtt{U}(\mathfrak{g}) \end{array}$$

Considering $\mathtt{U}(\mathfrak{g})$ as a subalgebra of $\mathtt{U}(\mathfrak{h} \rtimes \mathfrak{g})$, there are induced algebra actions of $\mathtt{U}(\mathfrak{g})$ on $\mathtt{U}(\mathfrak{h} \rtimes \mathfrak{g})$ and on $\mathtt{U}(\mathfrak{h} \rtimes \mathfrak{g}) \otimes (N \oplus M)$. There are also two morphisms

$$\xi_1\colon (\mathtt{U}(\mathfrak{g}) \otimes M) \otimes_{\mathtt{U}(\mathfrak{g})} \mathtt{U}(\mathfrak{h} \rtimes \mathfrak{g}) \to \mathtt{U}(\mathfrak{h} \rtimes \mathfrak{g}) \otimes (N \oplus M),$$

$$\xi_2\colon \mathtt{U}(\mathfrak{h} \rtimes \mathfrak{g}) \otimes_{\mathtt{U}(\mathfrak{g})} (\mathtt{U}(\mathfrak{g}) \otimes M) \to \mathtt{U}(\mathfrak{h} \rtimes \mathfrak{g}) \otimes (N \oplus M),$$

where $\xi_1\big((1 \otimes m), (h,g)\big) = (h,g) \otimes (0,m) + 1 \otimes (\xi(m,h), [m,g])$ and $\xi_2\big((h,g), (1 \otimes m)\big) = (h,g) \otimes (0,m)$, where $\xi\colon M \otimes \mathfrak{h} \to N$ is the homomorphism from the action of $(M, \mathfrak{g})$ on $(N, \mathfrak{h})$. These actions and homomorphisms define an action of algebras in $\mathcal{LM}$.

Let us consider the ideal of $\mathtt{U}(\mathfrak{h} \rtimes \mathfrak{g}) \otimes (N \oplus M)$ given by $\mathcal{Y}' = \operatorname{Ker}\mathtt{U}(s_1)\operatorname{Ker}\mathtt{U}(t_2) + \operatorname{Ker}\mathtt{U}(s_2)\operatorname{Ker}\mathtt{U}(t_1) + \operatorname{Ker}\mathtt{U}(t_1)\operatorname{Ker}\mathtt{U}(s_2) + \operatorname{Ker}\mathtt{U}(t_2)\operatorname{Ker}\mathtt{U}(s_1)$ and the ideal of $\mathtt{U}(\mathfrak{h} \rtimes \mathfrak{g})$ given by $\mathcal{X}' = \operatorname{Ker}\mathtt{U}(s_2)\operatorname{Ker}\mathtt{U}(t_2) + \operatorname{Ker}\mathtt{U}(t_2)\operatorname{Ker}\mathtt{U}(s_2)$.

Lemma 7.6.4. *The two squares*

$$\begin{array}{ccc}
\dfrac{\mathtt{U}(\mathfrak{h} \rtimes \mathfrak{g}) \otimes (N \oplus M)}{\mathcal{Y}'} & \overset{\overline{\mathtt{U}}(s_1)}{\underset{\overline{\mathtt{U}}(t_1)}{\rightrightarrows}} & \mathtt{U}(\mathfrak{g}) \otimes M \\
\Big\downarrow{\scriptstyle \overline{\mathtt{U}}(\beta\oplus\alpha)} & & \Big\downarrow{\scriptstyle \mathtt{U}(\alpha)} \\
\dfrac{\mathtt{U}(\mathfrak{h} \rtimes \mathfrak{g})}{\mathcal{X}'} & \overset{\overline{\mathtt{U}}(s_2)}{\underset{\overline{\mathtt{U}}(t_2)}{\rightrightarrows}} & \mathtt{U}(\mathfrak{g})
\end{array}$$

are well defined. Moreover, the actions of $\mathtt{U}(\mathfrak{g})$ on $\mathtt{U}(\mathfrak{h} \rtimes \mathfrak{g})$ and $\mathtt{U}(\mathfrak{h} \rtimes \mathfrak{g})\otimes(N\oplus M)$ and the morphisms ξ_1 and ξ_2 factor through $\mathcal{X}'$ and $\mathcal{Y}'$.

Proof. Since the bottom row is the Lie algebra case, the proof can be found in [13]. The top row follows by definition of $\mathcal{Y}'$. The homomorphism $\mathtt{U}(\beta \oplus \alpha)$ is zero in $\mathcal{Y}'$ by the equivariance of $\alpha \oplus \beta$ and the commutativity of the diagram. Again, the action of $\mathtt{U}(\mathfrak{g})$ on $\mathcal{X}'$ is zero since we are exactly in the Lie case. It is obvious that $\mathcal{X}'$ acts trivially on $\mathtt{U}(\mathfrak{h} \rtimes \mathfrak{g}) \otimes (N \oplus M)$ and that $\mathtt{U}(\mathfrak{h} \rtimes \mathfrak{g})$, and consequently $\mathtt{U}(\mathfrak{g})$, acts trivially on $\mathcal{Y}'$. Let $1 \otimes m \in \mathtt{U}(\mathfrak{g}) \otimes M$. Then $\xi_2\big(\operatorname{Ker}\mathtt{U}(s_2)\operatorname{Ker}\mathtt{U}(t_2), 1 \otimes m\big) = \operatorname{Ker}\mathtt{U}(s_2)\operatorname{Ker}\mathtt{U}(t_2) \otimes (0, m) \subseteq \mathcal{Y}'$ and the same happens for $\operatorname{Ker}\mathtt{U}(t_2)\operatorname{Ker}\mathtt{U}(s_2)$. On the other hand, $\xi_1\big(1 \otimes m, \operatorname{Ker}\mathtt{U}(s_2)\operatorname{Ker}\mathtt{U}(t_2)\big)$ is equal to $\operatorname{Ker}\mathtt{U}(s_2)\operatorname{Ker}\mathtt{U}(t_2) \otimes (0, m) \subseteq \mathcal{Y}'$ plus $\operatorname{Ker}\mathtt{U}(s_2)\operatorname{Ker}\mathtt{U}(t_2)$ acting on $1 \otimes (0, m)$, which clearly is also inside $\mathcal{Y}'$. □

We consider now $\big(\operatorname{Ker}\overline{\mathtt{U}}(s_1), \operatorname{Ker}\overline{\mathtt{U}}(s_2)\big)$. The restriction of $\big(\overline{\mathtt{U}}(t_1), \overline{\mathtt{U}}(t_2)\big)$ and $\overline{\mathtt{U}}(\beta \oplus \alpha)$ to this kernel will be denoted in the same way by abuse of notation. Then we have the following result.

Theorem 7.6.5. *Let $(\varrho_1, \varrho_2)\colon (N, \mathfrak{h}) \to (M, \mathfrak{g})$ be a Lie crossed module in $\mathcal{LM}$. The following square illustrates a crossed module of algebras in $\mathcal{LM}$ with the induced actions*

$$\begin{array}{ccc}
\operatorname{Ker}\overline{\mathtt{U}}(s_1) & \xrightarrow{\overline{\mathtt{U}}(t_1)} & \mathtt{U}(\mathfrak{g}) \otimes M \\
\Big\downarrow{\scriptstyle \overline{\mathtt{U}}(\beta\oplus\alpha)} & & \Big\downarrow{\scriptstyle \mathtt{U}(\alpha)} \\
\operatorname{Ker}\overline{\mathtt{U}}(s_2) & \xrightarrow{\overline{\mathtt{U}}(t_2)} & \mathtt{U}(\mathfrak{g})
\end{array}$$

Moreover, it is the universal enveloping crossed module of algebras of $(\varrho_1, \varrho_2)\colon (N, \mathfrak{h}) \to (M, \mathfrak{g})$ *in* $\mathcal{LM}$.

Proof. The action is studied in Lemma 7.6.4, and the bottom row is a crossed module of algebras since it is just the Lie algebra case. The second condition follows straightforwardly from definitions of ξ_1 and ξ_2. □

Let us now consider a crossed module of Leibniz algebras $(\mathfrak{q}, \mathfrak{p}, \eta)$. It can be viewed as a crossed module of Lie algebras in $\mathcal{LM}$ as a square

$$\begin{array}{ccc} \mathfrak{q} & \xrightarrow{\eta} & \mathfrak{p} \\ \downarrow & & \downarrow \\ \dfrac{\mathtt{Lie}(\mathfrak{q})}{[\mathfrak{q}, \mathfrak{p}]_{\mathtt{x}}} & \xrightarrow{\overline{\eta}} & \mathtt{Lie}(\mathfrak{p}) \end{array}$$

We can consider its universal enveloping algebra, obtaining that way a crossed module of algebras in $\mathcal{LM}$

$$\begin{array}{ccc} \operatorname{Ker} \overline{\mathtt{U}}(s_1) & \xrightarrow{\overline{\mathtt{U}}(t_1)} & \mathtt{U}\big(\mathtt{Lie}(\mathfrak{p})\big) \otimes \mathfrak{p} \\ \downarrow & & \downarrow \\ \operatorname{Ker} \overline{\mathtt{U}}(s_2) & \xrightarrow{\overline{\mathtt{U}}(t_2)} & \mathtt{U}\big(\mathtt{Lie}(\mathfrak{p})\big) \end{array}$$

Following the construction (7.6.1) we obtain its corresponding crossed module of associative algebras in the classical setting

$$\Big(\operatorname{Ker} \overline{\mathtt{U}}(s_1) \oplus \operatorname{Ker} \overline{\mathtt{U}}(s_2), \big(\mathtt{U}(\mathtt{Lie}(\mathfrak{p})) \otimes \mathfrak{p} \big) \oplus \mathtt{U}(\mathtt{Lie}(\mathfrak{p})), \big(\overline{\mathtt{U}}(t_1), \overline{\mathtt{U}}(t_2)\big) \Big).$$

Theorem 7.6.6. *Let* $(\mathfrak{q}, \mathfrak{p}, \eta)$ *be a crossed module of Leibniz algebras. Its universal enveloping crossed module of algebras* $\mathit{XUL}(\mathfrak{q}, \mathfrak{p}, \eta)$ *is isomorphic to the crossed module of algebras defined above.*

Proof. We have the following diagram

$$\begin{array}{ccc} \mathtt{U}\big(\mathtt{Lie}(\mathfrak{q} \rtimes \mathfrak{p})\big) \otimes (\mathfrak{q} \rtimes \mathfrak{p}) & \xrightarrow{\overline{\mathtt{U}}(t_1)} & \mathtt{U}\big(\mathtt{Lie}(\mathfrak{p})\big) \otimes \mathfrak{p} \\ \downarrow & & \downarrow \\ \mathtt{U}\big(\mathtt{Lie}(\mathfrak{q} \rtimes \mathfrak{p})\big) & \xrightarrow{\overline{\mathtt{U}}(t_2)} & \mathtt{U}\big(\mathtt{Lie}(\mathfrak{p})\big) \end{array}$$

By [21, (2.4) Proposition] we know that the direct sum of the two objects of the first column is isomorphic to $\mathtt{UL}(\mathfrak{q} \rtimes \mathfrak{p})$ and the direct sum of the objects of the second column is isomorphic to $\mathtt{UL}(\mathfrak{p})$. Let $\mathcal{X}$ be the ideal $\operatorname{Ker}\mathtt{UL}(s)\operatorname{Ker}\mathtt{UL}(t)+\operatorname{Ker}\mathtt{UL}(t)\operatorname{Ker}\mathtt{UL}(s)$ defined in Section 7.4. Consider the isomorphism

$$\theta\colon \mathtt{UL}(\mathfrak{q} \rtimes \mathfrak{p}) \xrightarrow{\sim} \left(\mathtt{U}\left(\frac{\mathtt{Lie}(\mathfrak{q})}{[\mathfrak{q},\mathfrak{p}]_{\mathtt{x}}} \rtimes \mathtt{Lie}(\mathfrak{p})\right) \otimes (\mathfrak{q} \rtimes \mathfrak{p})\right) \oplus \mathtt{U}\left(\frac{\mathtt{Lie}(\mathfrak{q})}{[\mathfrak{q},\mathfrak{p}]_{\mathtt{x}}} \rtimes \mathtt{Lie}(\mathfrak{p})\right)$$

defined on generators by $\theta\big((q,p)_r\big) = (\overline{q},\overline{p})$ and $\theta\big((q,p)_l\big) = 1 \otimes (\overline{q},\overline{p})$. We need to check that it maps $\mathcal{X}$ to $\mathcal{X}' + \mathcal{Y}'$ and θ^{-1} maps $\mathcal{X}' + \mathcal{Y}'$ to $\mathcal{X}$, but this follows straightforwardly by the definitions of $\mathcal{X}$, $\mathcal{X}'$ and $\mathcal{Y}'$ completing the proof. □

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Chapter 8

Do n-Lie algebras have universal enveloping algebras?

Abstract

The aim of this paper is to investigate in which sense, for $n \geqslant 3$, n-Lie algebras admit universal enveloping algebras. There have been some attempts at a construction (see [11] and [5]) but after analysing those we come to the conclusion that they cannot be valid in general. We give counterexamples and sufficient conditions.

We then study the problem in its full generality, showing that universality is incompatible with the wish that the category of modules over a given n-Lie algebra L is equivalent to the category of modules over the associated algebra $\mathrm{U}(L)$. Indeed, an *associated algebra functor* $\mathrm{U}\colon n\text{-}\mathsf{Lie}_{\mathbb{K}} \to \mathsf{Alg}_{\mathbb{K}}$ inducing such an equivalence does exist, but this kind of functor never admits a right adjoint.

We end the paper by introducing a (co)homology theory based on the associated algebra functor U.

Reference

X. García-Martínez, R. Turdibaev, and T. Van der Linden, *Do n-Lie algebras have universal enveloping algebras*, J. Lie Theory **28** (2018), no. 1, 43–55.

8.1 Introduction

The algebraic concept of an *n-Lie algebra* (also called a *Filippov algebra* or a *Nambu algebra*) is a natural generalisation of Lie algebras. Alternative generalisations of Lie algebras to n-ary brackets exist, such as *Lie triple systems* [17], but we shall not study those in the present paper. By definition, an n-Lie algebra is a $\mathbb{K}$-module with a skew-symmetric n-ary operation which is also a derivation. In recent years these have shown their relevance in some areas of physics such as Nambu mechanics [20] or string and membrane theory [2, 3].

In this article we investigate how to extend the concept of *universal enveloping algebra*, an important basic tool in theory of ordinary (= 2-) Lie algebras, to n-Lie algebras where $n \geqslant 3$.

Given a Lie algebra L, its universal enveloping algebra $\mathrm{U}(L)$ has three distinguishing characteristics:

(U1) *equivalent representations*: the category of Lie modules over L is equivalent to the category of "standard" modules over $\mathrm{U}(L)$;

(U2) *universality*: the functor $\mathrm{U}\colon \mathsf{Lie}_{\mathbb{K}} \to \mathsf{Alg}_{\mathbb{K}}$ has a right adjoint

$$(-)_{\mathrm{Lie}}\colon \mathsf{Alg}_{\mathbb{K}} \to \mathsf{Lie}_{\mathbb{K}}$$

which endows an associative algebra with a Lie algebra structure via the bracket $[a, b] = ab - ba$;

(U3) enveloping algebras are *enveloping*: if L is free as $\mathbb{K}$-module (for instance, whenever $\mathbb{K}$ is a field), then the L-component $\eta_L\colon L \to \mathrm{U}(L)$ of the unit η of the adjunction considered in (U2) is a monomorphism [16].

In the literature, already some attempts at introducing universal enveloping algebras for n-Lie algebras have been made [5, 11]. However, in the beginning of Section 8.3 we give an example showing that those cannot be fully valid. The problem with these approaches is that they depend on the existence of a functor from n-$\mathsf{Lie}_{\mathbb{K}}$ to $\mathsf{Lie}_{\mathbb{K}}$, analogous to the Daletskii-Takhtajan functor for Leibniz algebras [10]. The construction proposed in [5], though, produces an object is not always a Lie algebra. That is to say, the "functor" in question does not land in the right category. In Corollary 8.3.3 and Proposition 8.3.6 we give some conditions which establish when the construction of [5] is, or isn't, a Lie algebra. Luckily, this imprecise definition is not an obstruction to further results in the papers [5, 11], since those focus on simple n-Lie algebras over the complex numbers, and in Remark 8.3.10 we explain that for those n-Lie algebras the construction proposed in [5, 11] does indeed work. Nevertheless, the general definition of universal enveloping algebra proposed there is not correct.

Another point of view in this topic was given by Elgendy and Bremner in [12], where they study the universal enveloping algebra of an n-Lie algebra in an alternative setting. Although it does not give us information about our problem, this framework is also interesting and challenging.

One problem we face when extending the concept of universal enveloping algebra to the category of n-Lie algebras is the lack of a natural generalisation of the functor $(-)_{\mathrm{Lie}}$, so that U cannot be defined via (U2). Therefore, using a standard categorical technique, in Section 8.4 we define a functor $\mathrm{U}\colon n\text{-}\mathsf{Lie}_{\mathbb{K}} \to \mathsf{Alg}_{\mathbb{K}}$ such that (U1) holds: the category of modules over an n-Lie algebra L is equivalent to the category of $\mathrm{U}(L)$-modules. It happens that this functor does not have a right adjoint. In fact, we prove that *any* functor satisfying (U1) cannot have a right adjoint of the kind needed for (U2), so that the requirements (U1) and (U2) are shown to be mutually incompatible. And without condition (U2), the third requirement (U3), which asks that components of the unit of the adjunction from (U2) are monomorphisms, loses its sense. We thus end up with a functor $\mathrm{U}\colon n\text{-}\mathsf{Lie}_{\mathbb{K}} \to \mathsf{Alg}_{\mathbb{K}}$ satisfying just (U1), which we call the *associated algebra functor*.

In the final Section 8.5 we extend Lie algebra (co)homology to a (co)homology theory based on this associated algebra functor and we prove it to be different from the cohomology theories introduced in [23], [10] and [1].

8.2 Preliminaries on n-Lie algebras

Let $\mathbb{K}$ be a commutative unital ring and n a natural number, $n \geqslant 2$. The following definitions first appeared in [13, 8].

8.2.1 n-Leibniz algebras and n-Lie algebras

An **n-Leibniz algebra** L is a $\mathbb{K}$-module equipped with an n-linear operation $L^n \to L$, so a linear map $[-,\ldots,-]\colon L^{\otimes n} \to L$, satisfying the identity

$$\big[[x_1,\ldots,x_n],y_1,\ldots,y_{n-1}\big] = \sum_{i=1}^{n} \big[x_1,\ldots,x_{i-1},[x_i,y_1,\ldots,y_n],x_{i+1},\ldots,x_n\big] \tag{$\star$}$$

for all x_i, $y_i \in L$. A homomorphism of n-Leibniz algebras is a $\mathbb{K}$-module homomorphism preserving this bracket; this defines the category $n\text{-}\mathsf{Leib}_{\mathbb{K}}$.

An **n-Lie algebra** is an n-Leibniz algebra L where the bracket $[-,\ldots,-]$ factors through the exterior product to a morphism

$$\Lambda^n L = \underbrace{L \wedge \cdots \wedge L}_{n \text{ factors}} \to L.$$

We thus obtain the full subcategory $n\text{-}\mathsf{Lie}_{\mathbb{K}}$ of $n\text{-}\mathsf{Leib}_{\mathbb{K}}$ determined by the n-Lie algebras.

The latter condition means that the bracket $[-,\ldots,-]$ is not just n-linear, but also **alternating**: it vanishes on any n-tuple with a pair of equal coordinates. In other words, $[x_1,\ldots,x_n] = 0$ as soon as there exist $1 \leqslant i < j \leqslant n$ for which $x_i = x_j$.

When $n = 2$, identity $(\star)$ yields the Leibniz identity. In this case, being alternating is equivalent to skew-symmetry, which gives the Jacobi identity. Thus the above definition describes Leibniz and Lie algebras, respectively.

8.2.2 Derivations

A linear endomap $d\colon L \to L$ on an n-Lie algebra L is called a **derivation** if

$$d([x_1,x_2,\ldots,x_n]) = \sum_{i=1}^{n} [x_1,\ldots,d(x_i),\ldots,x_n].$$

The $\mathbb{K}$-module of all derivations of a given n-Lie algebra L is denoted by $\mathrm{Der}(L)$ and forms a Lie algebra with respect to the commutator $[d_1,d_2] = d_1d_2 - d_2d_1$.

8.2.3 Ideals

An **ideal** of an n-Lie algebra is a normal subalgebra. It is easily seen that a $\mathbb{K}$-submodule I of an n-Lie algebra L is an ideal if and only if $[I,L,\ldots,L] \subseteq I$.

8.2.4 Right multiplication, adjoint action

Given a generator $x = x_1 \otimes \cdots \otimes x_{n-1}$ of $L^{\otimes(n-1)}$, the **right multiplication** and the **adjoint action** (also called **left multiplication**) by x are maps

$$R_x = R(x_1,\ldots,x_{n-1}) \quad \text{and} \quad \mathrm{ad}_x = \mathrm{ad}(x_1,\ldots,x_{n-1})\colon L \to L$$

respectively defined by

$$R(x_1, \ldots, x_{n-1})(a) = [a, x_1, \ldots, x_{n-1}]$$

and

$$\mathrm{ad}(x_1, \ldots, x_{n-1})(b) = [x_1, \ldots, x_{n-1}, b]$$

for a, $b \in L$. Clearly, $\mathrm{ad}_x = (-1)^{n-1} R_x$, and due to identity ($\star$) both maps are derivations. They are called **inner derivations** of L and generate an ideal $\mathrm{InnDer}(L)$ of $\mathrm{Der}(L)$. We will use the same notations ad_x and R_x for the extensions (by derivation) of these maps to the entire tensor algebra $T(L)$. (That is to say, $R_x(a_1 \otimes a_2) = R_x(a_1) \otimes a_2 + a_1 \otimes R_x(a_2)$, etc.)

8.2.5 The centre

The ideal $Z(L) = \{z \in L \mid \mathrm{ad}_x(z) = 0, \forall x \in L^{\otimes(n-1)}\}$ is called the **centre** of L.

8.2.6 Simple n-Lie algebras

Given an ideal I, we write $I^1 = [I, L, \ldots, L]$ for the $\mathbb{K}$-submodule spanned by the elements $R_x(i)$ where $i \in I$ and $x \in L^{\otimes(n-1)}$. It is easy to see that I^1 is an ideal of L. If $L^1 \neq 0$ (so that it is non-abelian, i.e., it doesn't come equipped with the zero bracket) and L does not admit any non-trivial ideals then L is called a **simple** n-Lie algebra.

We now recall from [13] an important example of an $(n+1)$-dimensional n-Lie algebra which is an analogue of the three-dimensional Lie algebra with the cross product as multiplication.

Example 8.2.1. Let $\mathbb{K}$ be a field and V_n an $(n+1)$-dimensional $\mathbb{K}$-vector space with a basis $\{e_1, \ldots, e_{n+1}\}$. Then V_n, equipped with the skew-symmetric n-ary multiplication induced by

$$[e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_{n+1}] = (-1)^{n+1+i} e_i, \qquad 1 \leqslant i \leqslant n+1,$$

is an n-Lie algebra.

This algebra is a simple n-Lie algebra. Conversely, as shown in [18], over an algebraically closed field $\mathbb{K}$ all simple n-Lie algebras are isomorphic to V_n.

8.2.7 Leibniz and Lie algebras associated to an n-Lie algebra

Given an n-Lie algebra L, we introduce the operations

$$[-,-]\colon L^{\otimes(2n-2)} \to L^{\otimes(n-1)}\colon x \otimes y \mapsto [x,y] = \mathrm{ad}_x(y),$$
$$-\circ -\colon L^{\otimes(2n-2)} \to L^{\otimes(n-1)}\colon x \otimes y \mapsto x \circ y = \tfrac{1}{2}\big(\mathrm{ad}_x(y) - \mathrm{ad}_y(x)\big).$$

Note that $\circ$ is skew-symmetric. Furthermore, the operations coincide if and only if $\mathrm{ad}_x(y) = -\mathrm{ad}_y(x)$, i.e., when $[-,-]$ is skew-symmetric. These two products have the following property relating them to the adjoint action.

Proposition 8.2.2. *For any x, $y \in L^{\otimes(n-1)}$ the equality*

$$[\mathrm{ad}_x, \mathrm{ad}_y] = \mathrm{ad}_{[x,y]} = \mathrm{ad}_{x\circ y}$$

holds.

Proof. Let $x = x_2 \otimes \cdots \otimes x_n$, $y = y_2 \otimes \cdots \otimes y_n$ and $x_1 \in L$. Then from identity $(\star)$ we deduce

$$\mathrm{ad}_y \mathrm{ad}_x(x_1) = \mathrm{ad}_x \mathrm{ad}_y(x_1) + (-1)^{n-1} \sum_{k=2}^{n} [x_1, \dots, \mathrm{ad}_y(x_k), \dots, x_n],$$

which is equivalent to

$$[\mathrm{ad}_x, \mathrm{ad}_y](x_1) = (-1)^n \sum_{k=2}^{n} [x_1, x_2, \dots, \mathrm{ad}_y(x_k), \dots, x_n] = -\mathrm{ad}_z(x_1),$$

where $z = \mathrm{ad}_y(x)$. By symmetry, $[\mathrm{ad}_y, \mathrm{ad}_x] = -\mathrm{ad}_w$, where $w = \mathrm{ad}_x(y)$. Since $\mathrm{InnDer}(L)$ is a Lie algebra we obtain $[\mathrm{ad}_x, \mathrm{ad}_y] = \mathrm{ad}_w = \mathrm{ad}_{[x,y]}$ and $[\mathrm{ad}_x, \mathrm{ad}_y] = \frac{1}{2}(\mathrm{ad}_w - \mathrm{ad}_v) = \mathrm{ad}_{x\circ y}$. ☐

The following result is due to Daletskii and Takhtajan [10].

Theorem 8.2.3. *Let L be an n-Lie algebra. Then $L^{\otimes(n-1)}$ with bracket $[-,-]$ is a Leibniz algebra.* ☐

This algebra is called the **basic Leibniz algebra** associated to an n-Lie algebra L. We denote it by $\mathrm{BLb}^{\otimes}_{n-1}(L)$.

Following [10] let us write $\mathcal{K}_{n-1} = \mathrm{Span}\{x \in L^{\otimes(n-1)} \mid \mathrm{ad}_x = 0\}$ for the kernel of the adjoint action. We recall the following result from [10].

Theorem 8.2.4. *The subspace $\mathcal{K}_{n-1}$ is an ideal of* $\mathrm{BLb}_{n-1}^{\otimes}(L)$ *and the quotient algebra* $\mathrm{BLb}_{n-1}^{\otimes}(L)/\mathcal{K}_{n-1}$ *is a Lie algebra.* □

This Lie algebra was introduced in [9] and called the **basic Lie algebra** of the given n-Lie algebra L.

8.3 Algebras associated to an n-Lie algebra

Given an n-Lie algebra L over the complex numbers $\mathbb{C}$, in the article [5] the authors consider the algebra $(\Lambda^{n-1}L, \circ)$. In Proposition 1 of [5], this product $\circ$ is claimed to satisfy the Jacobi identity. However, this cannot be correct, as we may see in the following example of a 3-Lie algebra, which is a member of the class of so-called *filiform* 3-Lie algebras given in [15]. For the sake of simplicity let us denote $\mathrm{J}(a,b,c) = a \circ (b \circ c) + c \circ (a \circ b) + b \circ (c \circ a)$.

Example 8.3.1. Consider the 3-Lie algebra with basis $\{x_1, x_2, x_3, x_4, x_5\}$ and the table of multiplication determined by

$$[x_1, x_2, x_3] = x_4, \qquad [x_1, x_2, x_4] = [x_1, x_3, x_4] = [x_2, x_3, x_4] = x_5.$$

Then $\mathrm{J}(x_1 \wedge x_4, x_1 \wedge x_2, x_3 \wedge x_2) = -\frac{1}{4} x_4 \wedge x_5 \neq 0$.

In order to determine when the algebra $(\Lambda^{n-1}L, \circ)$ defined in [5] is actually a Lie algebra, let us have a look at the terms of Jacobi identity.

Proposition 8.3.2. *Let L be an n-Lie algebra. Then for any a, b, $c \in \Lambda^{n-1}L$ the following equality holds:*

$$\mathrm{J}(a,b,c) = -\tfrac{1}{4}\big([\mathrm{ad}_b, \mathrm{ad}_c](a) + [\mathrm{ad}_a, \mathrm{ad}_b](c) + [\mathrm{ad}_c, \mathrm{ad}_a](b)\big) \tag{†}$$

Proof. By Proposition 8.2.2 we obtain

$$\begin{aligned} a \circ \big(b \circ c\big) &= \tfrac{1}{2}\big(\mathrm{ad}_a(b \circ c) - \mathrm{ad}_{b\circ c}(a)\big) \\ &= \tfrac{1}{2}\left(\mathrm{ad}_a\big(\tfrac{1}{2}\big(\mathrm{ad}_b(c) - \mathrm{ad}_c(b)\big)\big)\right) - \tfrac{1}{2}\,\mathrm{ad}_{b\circ c}(a) \\ &= \tfrac{1}{4}\,\mathrm{ad}_a\big(\mathrm{ad}_b(c)\big) - \tfrac{1}{4}\,\mathrm{ad}_a\big(\mathrm{ad}_c(b)\big) - \tfrac{1}{2}[\mathrm{ad}_b, \mathrm{ad}_c](a). \end{aligned}$$

After similar calculations for the other terms, equality (†) follows. □

Corollary 8.3.3. *Let L be an n-Lie algebra with abelian* $\mathrm{InnDer}(L)$. *Then* $(\Lambda^{n-1}L, \circ)$ *is a Lie algebra.* □

Remark 8.3.4. Corollary 8.3.3 provides us with a sufficient condition. Due to the results in [21], for $n = 3$ it also seems to be necessary. Indeed, while considering a more general question, in that work a similar product appears. Now given a free skew-symmetric ternary algebra $(F, [-,-,-])$, the authors of [21] consider $F \wedge F$ equipped with the product $x \cdot y = \mathrm{ad}_y(x) - \mathrm{ad}_x(y) = -2(x \circ y)$. Observe that $(F \wedge F, \cdot)$ is a Lie algebra if and only if $(F \wedge F, \circ)$ is a Lie algebra.

Remark 8.3.5. It is claimed in [21, Theorem 3.1] that if I is a non-zero minimal ideal of F such that quotient $F/I \wedge F/I$ is a Lie algebra then

$$I = \langle [[x_1, x_2, x_3], x_4, x_5] - [[x_1, x_4, x_5], x_2, x_3] \mid x_i \in F \rangle.$$

However, this result cannot be correct. Indeed, consider the central extension $F = \mathbb{C}z \oplus V_3$ of the simple 3-Lie algebra V_3 over $\mathbb{C}$ from Example 8.2.1. We have $I = \langle [\mathrm{ad}_x, \mathrm{ad}_y](a) \mid a \in F,\ x,\ y \in F \wedge F \rangle \subseteq V_3$ so that $I = V_3$ because V_3 is simple. Set $a = e_2 \wedge e_4$, $b = e_1 \wedge e_2$, $c = e_1 \wedge z$ and observe that $\mathrm{ad}_c = 0$, $[\mathrm{ad}_a, \mathrm{ad}_b](e_1) = e_4$, which yields $\mathrm{J}(a, b, c) = z \wedge e_4 \neq 0$. Hence $(F \wedge F, \cdot)$ is not a Lie algebra. However, by [4, Corollary 1.2.4], $(V_3 \wedge V_3, \circ)$ is indeed a Lie algebra (isomorphic to $\mathfrak{so}_4$). As a consequence, $Z(F) = \mathbb{C}z$ is another minimal ideal with the property that the quotient algebra is a Lie algebra.

It follows that the condition of Corollary 8.3.3 is sufficient, but not necessary. A precise characterisation seems hard to find, but we have the following partial result.

Proposition 8.3.6. *Let $\mathbb{K}$ be a field and let L be a 3-Lie algebra over $\mathbb{K}$ such that* $\mathrm{InnDer}(L)$ *is not abelian. If* $\dim Z(L) \geqslant 2$ *then* $(\Lambda^{n-1}L, \circ)$ *is not a Lie algebra.*

Proof. By assumption there are some $x, y \in L \wedge L$ with $[\mathrm{ad}_x, \mathrm{ad}_y] = \mathrm{ad}_{[x,y]} \neq 0$. Pick an element $z_1 \in Z(L)$ and, if possible, take $z_2 \in L$ such that $\mathrm{ad}_{[x,y]}(z_2) \notin \mathrm{Span}\{z_1\}$. In other words, $z_1 \wedge \mathrm{ad}_{[x,y]}(z_2) \neq 0$. Putting $z = z_1 \wedge z_2$ yields $\mathrm{ad}_z(L) = 0$ and $\mathrm{ad}_{[y,z]}(x) + \mathrm{ad}_{[z,x]}(y) = 0$. However, $[\mathrm{ad}_x, \mathrm{ad}_y](z) = z_1 \wedge \mathrm{ad}_{[x,y]}(z_2) \neq 0$ and thus the Jacobi identity does not hold.

If such a z_2 does not exist, then let us assume that $\mathrm{ad}_{[x,y]}(L) = \mathrm{Span}\{z_1\}$. In this case, pick $z_3 \in Z(L)$ linearly independent from z_1. Choose a $z_4 \in L$ such that $\mathrm{ad}_{[x,y]}(z_4) = z_1$ and consider $z = z_3 \wedge z_4$. Obviously, $z \neq 0$ and $-4\,\mathrm{J}(x, y, z) = z_3 \wedge z_1$ which is not zero. □

In the remaining cases it is not clear whether $(\Lambda^{n-1}L, \circ)$ is a Lie algebra or not.

8.3.1 The basic Leibniz algebra $\mathrm{BLb}^{\Lambda}_{n-1}(L)$

It is hard to endow $\Lambda^{n-1}L$ with a Lie algebra structure but it inherits a Leibniz algebra structure from the basic Leibniz algebra of [10]. Consider the subspace

$$\mathcal{W}_{n-1} = \mathrm{Span}\{x_1 \otimes \cdots \otimes x_{n-1} \mid x_i = x_j \text{ for some } 1 \leqslant i < j \leqslant n-1\}$$

of $L^{\otimes(n-1)}$.

Proposition 8.3.7. *Let L be an n-Lie algebra. Then $\mathcal{W}_{n-1}$ is an ideal of $\mathrm{BLb}^{\otimes}_{n-1}(L)$ and $\Lambda^{n-1}L = \mathrm{BLb}^{\otimes}_{n-1}(L)/\mathcal{W}_{n-1}$.*

Proof. First, note that for any $w \in \mathcal{W}_{n-1}$ and $v \in L^{\otimes(n-1)}$ we have $[w,v] = \mathrm{ad}_w(v) = 0$, so that $\mathcal{W}_{n-1} \subseteq \mathcal{K}_{n-1}$. For $w = x_1 \otimes \cdots \otimes x_{n-1} \in \mathcal{W}_{n-1}$, where $x_i = x_j$ for some $1 \leqslant i < j \leqslant n-1$ and $v \in L^{\otimes(n-1)}$ we have

$$\begin{aligned}[v,w] = \mathrm{ad}_v(w) =& [x_1, \dots, \mathrm{ad}_v(x_i), \dots, x_j, \dots, x_{n-1}] \\ &+ [x_1, \dots, x_i, \dots, \mathrm{ad}_v(x_j), \dots, x_{n-1}] \\ &+ \sum_{k\neq i, k\neq j} [x_1, \dots, \mathrm{ad}_v(x_k), \dots, x_{n-1}] \in \mathcal{W}_{n-1}\end{aligned}$$

since the sum of the first two terms and every summand in the sum belongs to $\mathcal{W}_{n-1}$. Hence $\mathcal{W}_{n-1}$ is an ideal of the Leibniz algebra $\mathrm{BLb}^{\otimes}_{n-1}(L)$ and we may conclude that $\Lambda^{n-1}L = \mathrm{BLb}^{\otimes}_{n-1}(L)/\mathcal{W}_{n-1}$. □

Let us denote this Leibniz algebra $\left(\Lambda^{n-1}L, [-,-]\right)$ by $\mathrm{BLb}^{\Lambda}_{n-1}(L)$. The basic Lie algebra $\mathrm{BLb}^{\otimes}_{n-1}(L)/\mathcal{K}_{n-1}$ is a subalgebra of the Leibniz algebra $\mathrm{BLb}^{\Lambda}_{n-1}(L)$.

Remark 8.3.8. In [11], given an n-Lie algebra L, the vector space $\Lambda^{n-1}L$ is equipped with a product $[x,y] = R_x(y) = (-1)^{n-1}\,\mathrm{ad}_x(y)$. This algebra coincides with $\mathrm{BLb}^{\Lambda}_{n-1}(L)$ up to a sign $(-1)^{n-1}$ in the multiplications. This product is not skew-symmetric as shown in [4, Remark 1.1.16].

Proposition 8.3.9. $\mathrm{BLb}^{\Lambda}_{n-1}(L) \cong \mathrm{InnDer}(L)$ *if and only if* $\mathcal{K}_{n-1} = \mathcal{W}_{n-1}$.

Proof. Due to Proposition 8.2.2 and the n-Lie structure of L, the map

$$x = x_2 \wedge \cdots \wedge x_n \mapsto \mathrm{ad}_x$$

is a well-defined surjective Leibniz algebra homomorphism of $\mathrm{BLb}^{\Lambda}_{n-1}(L)$ onto $\mathrm{InnDer}(L)$. Now if the kernel of this map is zero, which means $\mathrm{ad}_x \neq 0$ for all $x \neq 0$, then this map is an isomorphism. □

Remark 8.3.10. Consider the simple n-Lie algebra V_n over $\mathbb{C}$ of Example 8.2.1. It is easily seen that $\mathcal{K}_{n-1} = \mathcal{W}_{n-1}$, so we have an isomorphism

$$\mathrm{BLb}^{\otimes}_{n-1}(V_n)/\mathcal{K}_{n-1} = \mathrm{BLb}^{\Lambda}_{n-1}(V_n) \cong \mathrm{InnDer}(V_n).$$

Moreover, by skew-symmetry of the bracket we have $x \circ y = [x, y]$ and therefore $(\Lambda^{n-1}L, \circ)$ is the same Lie algebra $\mathrm{InnDer}(V_n)$. A different construction of the simple n-Lie algebra is given in [5] and it is proven in [18] that its basic Lie algebra happens to be $\mathfrak{so}_{n+1}$ (see also [4, Corollary 1.2.4]). Hence, the algebras constructed in [11] and [5] for V_n coincide with the basic Lie algebra [10]

$$\mathrm{BLb}^{\otimes}_{n-1}(V_n)/\mathcal{K}_{n-1} = \mathrm{BLb}^{\Lambda}_{n-1}(V_n) = (\Lambda^{n-1}L, \circ) \cong \mathrm{InnDer}(V_n) \cong \mathfrak{so}_{n+1}.$$

We may conclude that, although the constructions of the papers [11] and [5] do not work in general, their results stay valid for the simple n-Lie algebra case. In [11] the finite-dimensional, irreducible representation of the simple n-Lie algebra is studied and in [5] irreducible highest weight representations of the same algebra are studied.

Remark 8.3.11. The recently published Erratum [6] states that Proposition 1 and the results of Section 2 of [5] are not generally correct unless the map

$$\widetilde{\mathrm{ad}} \colon \Lambda^{n-1}V \to \Lambda^{\bullet}V$$

is skew-symmetric. Using our notation (and correcting $\Lambda^{\bullet}V$ to $\mathrm{End}(\Lambda^{n-1}V)$) we assume that skew-symmetry of the homomorphism $\mathrm{ad} \colon \Lambda^{n-1}V \to \mathrm{End}(\Lambda^{n-1}V)$ means $\mathrm{ad}_x(y) = -\mathrm{ad}_y(x)$ for all $x,\ y \in \Lambda^{n-1}V$. This condition is equivalent to $a \circ b = \mathrm{ad}_a(b) = [a, b]$. The latter one forces $(\Lambda^{n-1}V, \circ)$ to coincide with $\mathrm{BLb}^{\Lambda}_{n-1}(V)$. Since one of them is antisymmetric and the other one is Leibniz algebra, the result is a Lie algebra.

8.4 The associated algebra construction

8.4.1 The category of modules over an n-Lie algebra

Following [7] and [19, Section II.6], given an n-Lie algebra L over $\mathbb{K}$, we say that the category of **L-modules** or **n-Lie modules over L** is $L\text{-}\mathsf{Mod}_{\mathbb{K}} = \mathsf{Ab}(n\text{-}\mathsf{Lie}_{\mathbb{K}} \downarrow L)$, the category of abelian group objects in the comma category $(n\text{-}\mathsf{Lie}_{\mathbb{K}} \downarrow L)$.

This definition may be unpacked as follows: an L-module is a $\mathbb{K}$-module M with a structure of n-Lie algebra on $M \oplus L$ such that L is a subalgebra of $M \oplus L$, M is an ideal of $M \oplus L$, and the bracket is zero if two elements are in M. A homomorphism of L-modules $f\colon M \to M'$ is determined by an n-Lie algebra homomorphism from $M \oplus L$ to $M' \oplus L$ which restricts to the identity on L. In the particular case of $n = 2$ we recover the notion of a Lie representation.

This may be further decompressed as follows. An L-module is a $\mathbb{K}$-module M with a linear map $[-,\ldots,-]\colon (\Lambda^{n-1}L) \otimes M \to M$ satisfying the relations

$$\begin{aligned}\big[x_1,\ldots,x_{n-1},[y_1,\ldots,y_{n-1},m]\big] - \big[y_1,\ldots,y_{n-1},[x_1,\ldots,x_{n-1},m]\big]\\ = \sum_{i=1}^{n-1}[y_1,\ldots,[x_1,\ldots,x_{n-1},y_i],\ldots,y_{n-1},m]\end{aligned}$$

and

$$\begin{aligned}\big[[x_1,\ldots,x_n],y_2,\ldots,y_{n-1},m\big] =\\ \sum_{i=1}^{n-1}(-1)^{n-i}\big[x_1,\ldots,\widehat{x}_i,\ldots,x_n,[x_i,y_2,\ldots,y_{n-1},m]\big],\end{aligned}$$

for all x_i, $y_i \in L$ and $m \in M$.

Example 8.4.1. The base ring $\mathbb{K}$ is an L-module via the trivial action.

8.4.2 The associated algebra functor

For any n-Lie algebra L, the category $L\text{-}\mathsf{Mod}_{\mathbb{K}}$ is an abelian variety of algebras. It is well-known that this makes it equivalent to the category of modules over the endomorphism algebra of the free L-module on one generator [14, page 106]. This process determines a functor $\mathrm{U}\colon n\text{-}\mathsf{Lie}_{\mathbb{K}} \to \mathsf{Alg}_{\mathbb{K}}$ from the category of n-Lie algebras to the category of associative unital $\mathbb{K}$-algebras such that $L\text{-}\mathsf{Mod}_{\mathbb{K}}$ is equivalent to the category $\mathsf{Mod}_{\mathrm{U}(L)}$ of "standard" modules over the associative algebra $\mathrm{U}(L)$. The following proposition gives an explicit algebraic description of the functor U.

Proposition 8.4.2. *Given an n-Lie algebra L, the algebra $\mathrm{U}(L)$ is the tensor algebra of $\Lambda^{n-1}L$ quotient by the two-sided ideal generated by*

$$(x_1 \wedge \cdots \wedge x_{n-1})(y_1 \wedge \cdots \wedge y_{n-1}) - (y_1 \wedge \cdots \wedge y_{n-1})(x_1 \wedge \cdots \wedge x_{n-1}) = \sum_{i=1}^{n-1} y_1 \wedge \cdots \wedge [x_1, \ldots, x_{n-1}, y_i] \wedge \cdots \wedge y_{n-1}$$

and

$$[x_1, \ldots, x_n] \wedge y_2 \wedge \cdots \wedge y_{n-1} = \sum_{i=1}^{n} (-1)^{n-i} (x_1 \wedge \cdots \wedge \widehat{x}_i \wedge \cdots \wedge x_n)(x_i \wedge y_2 \wedge \cdots \wedge y_{n-1}),$$

for x_i, $y_i \in L$ and $m \in M$. □

Note that when $n = 2$ we obtain the universal enveloping algebra of a Lie algebra. From the point of view of Proposition 8.4.2, the equivalence of categories $L\text{-}\mathsf{Mod}_{\mathbb{K}} \simeq \mathsf{Mod}_{\mathrm{U}(L)}$ may be recovered by using that the L-module bracket $[x_1, \ldots, x_{n-1}, m]$ defines a $\mathrm{U}(L)$-module action $(x_1 \wedge \cdots \wedge x_{n-1})m$ and vice versa.

Example 8.4.3. Let L_m be the free n-Lie algebra on m generators with $m < n - 1$. (An explicit description of the free n-Lie algebra can be found in [22]). Then $\mathrm{U}(L_m) = \mathbb{K}$, since the $(n-1)$st exterior product is zero.

Assume $m = n-1$. Then all brackets are zero and the relations of the associated algebra vanish straightforward. Hence $\mathrm{U}(L_m)$ is $\mathbb{K}[X]$, the commutative polynomial ring over $\mathbb{K}$ with one generator.

If $m \geqslant n$, we can forget the elements with brackets by the second relation in Proposition 8.4.2. Thus we see that

$$(n-2)(x_1 \wedge \cdots \wedge x_{n-1})(y_1 \wedge \cdots \wedge y_{n-1}) + (y_1 \wedge \cdots \wedge y_{n-1})(x_1 \wedge \cdots \wedge x_{n-1}) - \sum_{i=1}^{n-1} \left(\sum_{j=1}^{n-1} (-1)^{n-j+i} (x_1 \wedge \cdots \wedge \widehat{x}_j \wedge \cdots \wedge x_{n-1} \wedge y_i)(x_j \wedge y_1 \wedge \cdots \wedge \widehat{y}_i \wedge \cdots \wedge y_{n-1}) \right)$$

is zero.

In the case of $m = n$ this relation vanishes. Therefore $\mathrm{U}(L_n)$ is isomorphic to $\mathbb{K}\langle X_1, \ldots, X_n \rangle$, the non-commutative polynomial ring over $\mathbb{K}$ in n variables.

If $m > n$ then $\mathrm{U}(L_n)$ is the non-commutative polynomial ring on $\binom{m}{n}$ elements quotiented by the two-sided ideal generated by the above relation.

Example 8.4.4. Let L be an abelian n-Lie algebra with n generators. Then $\mathrm{U}(L) \cong \mathbb{K}[X_1, \ldots, X_{n-1}]$ since the first identity of the associated algebra makes it abelian, while the second one vanishes.

When $n = 2$ the functor U is *universal* in the sense that it has a right adjoint. Let us explain why this is not possible for general n.

Lemma 8.4.5. *Consider $n > 2$. If $F \colon n\text{-}\mathsf{Lie}_{\mathbb{K}} \to \mathsf{Alg}_{\mathbb{K}}$ preserves binary sums, then there is an n-Lie algebra L for which $F(L)$ is not Morita equivalent to $\mathrm{U}(L)$.*

Proof. Recall that if two rings (or, in particular, $\mathbb{K}$-algebras) are Morita equivalent, then their centres are isomorphic.

Let L_1 be the free n-Lie algebra generated by one element. The coproduct of $n-1$ copies of L_1 is the free n-Lie algebra on $n-1$ generators, denoted by L_{n-1}. Its associated algebra $\mathrm{U}(L_{n-1})$ is $\mathbb{K}[X]$ as in Example 8.4.3, whose centre is itself.

Now $\mathbb{K}[X]$ cannot be Morita equivalent to $F(L_1 + \cdots + L_1) \cong F(L_1) + \cdots + F(L_1)$: the latter algebra being a coproduct, its centre cannot be bigger than $\mathbb{K}$, so is strictly smaller than $\mathbb{K}[X]$. □

Theorem 8.4.6. *The functor $\mathrm{U} \colon n\text{-}\mathsf{Lie}_{\mathbb{K}} \to \mathsf{Alg}_{\mathbb{K}}$ has a right adjoint if and only if $n = 2$. More precisely, for $n > 2$ there is no functor $F \colon n\text{-}\mathsf{Lie}_{\mathbb{K}} \to \mathsf{Alg}_{\mathbb{K}}$ with a right adjoint $G \colon \mathsf{Alg}_{\mathbb{K}} \to n\text{-}\mathsf{Lie}_{\mathbb{K}}$ such that there is an equivalence of categories between $L\text{-}\mathsf{Mod}_{\mathbb{K}}$ and $\mathsf{Mod}_{F(L)}$ for all L.*

Proof. If $n = 2$ this result is well known. Consider the case when $n > 2$; assume that there is an adjoint pair $F \dashv G$ as required. Then, on the one hand, F preserves binary sums, while on the other hand, we have an equivalence of categories $L\text{-}\mathsf{Mod}_{\mathbb{K}} \simeq \mathsf{Mod}_{F(L)} \simeq \mathsf{Mod}_{\mathrm{U}(L)}$ for any n-Lie algebra L. This is in contradiction with Lemma 8.4.5. □

This theorem shows that for $n > 2$ there is no way we can obtain a functor $F \colon n\text{-}\mathsf{Lie}_{\mathbb{K}} \to \mathsf{Alg}_{\mathbb{K}}$ satisfying both requirements (U1) and (U2) of the introduction: to have an equivalence of categories between $L\text{-}\mathsf{Mod}_{\mathbb{K}}$ and $\mathsf{Mod}_{F(L)}$ for all L and to have a right adjoint for the functor F. In particular, it is shown that $\mathrm{U}(L)$ does not satisfy (U2). Of course there may still exist other functors F such that all $F(L)$ are Morita equivalent to $\mathrm{U}(L)$.

Remark 8.4.7. If $\mathbb{K} = \mathbb{C}$ and $L = V_n$, then $\mathrm{U}(L)$ coincides with the construction of the universal enveloping algebra given in [11] and [5]. Moreover, for any n-Lie algebra L such that $\mathrm{BLb}_{n-1}^{\Lambda}(L)$ is also a Lie algebra, $\mathrm{U}(L)$ is isomorphic to the universal enveloping algebra given in [11]. However, if $(\Lambda^{n-1}L, \circ)$ is a Lie algebra, then $\mathrm{U}(L)$ might be different from the universal enveloping algebra of [5].

8.5 (Co)Homology theory and the associated algebra

Let L be an n-Lie algebra and M an L-module. Let

$$M^L = \{m \in M \mid [x_1, \dots, x_{n-1}, m] = 0 \text{ for all } x_i \in L\}$$

be the **invariant submodule** of M, and let $M_L = M/LM$ be the **coinvariant submodule**. As in Lie algebras, we can obtain (co)homology theories deriving the invariants and coinvariants functors.

Definition 8.5.1. The **homology groups** of M **with coefficients in** L, denoted by $\mathrm{H}_*(L, M)$ are the left derived functors of $(-)_L$. The **cohomology groups** of M **with coefficients in** L, denoted by $\mathrm{H}^*(L, M)$ are the right derived functors of $(-)^L$.

There is an immediate relation between this (co)homology theory and the associated algebra. Let $\varepsilon\colon \mathrm{U}(L) \to \mathbb{K}$ be the $\mathbb{K}$-algebra homomorphism sending the inclusion of $\Lambda^{n-1}L$ to zero. Its kernel, $\Omega(L)$, is called the **augmentation ideal**. Therefore, $\Omega(L)$ has a $\mathrm{U}(L)$-module structure.

Proposition 8.5.2. *Let L be an n-Lie algebra and M an L-module. There are isomorphisms*

$$\mathrm{H}_*(L, M) \cong \mathrm{Tor}_*^{\mathrm{U}(L)}(\mathbb{K}, M),$$
$$\mathrm{H}^*(L, M) \cong \mathrm{Ext}^*_{\mathrm{U}(L)}(\mathbb{K}, M).$$

Proof. As in the Lie algebra case (see [24]), we just have to check that the underlying functors are the same.

$$\mathbb{K} \otimes_{\mathrm{U}(L)} M = \tfrac{\mathrm{U}(L)}{\Omega(L)} \otimes_{\mathrm{U}(L)} M \cong \tfrac{M}{\Omega(L)M} = \tfrac{M}{LM} = M_L,$$

and

$$\mathrm{Hom}_{\mathrm{U}(L)}(\mathbb{K}, M) = \mathrm{Hom}_L(\mathbb{K}, M) = M^L. \qquad \square$$

Following the computations done for Lie algebras in [24, Section 7.4] we obtain that $\mathrm{H}_1(L,\mathbb{K}) \cong \Omega(L)/\Omega(L)^2$ and $\mathrm{H}^1(L,\mathbb{K}) \cong \mathrm{Hom}_{\mathbb{K}}(\Omega(L),\mathbb{K})$. In the particular case of Example 8.4.3, we see that

$$\mathrm{H}_1(L_m,\mathbb{K}) \cong \coprod_{\binom{m}{n}} \mathbb{K} \qquad \text{and} \qquad \mathrm{H}^1(L_m,\mathbb{K}) \cong \prod_{\binom{m}{n}} \mathbb{K}.$$

These results show that the cohomology theory defined above is different from the n-Lie algebra cohomology theories studied in [23], [10] and [1] when $n > 2$.

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Chapter 9

A new characterisation of groups amongst monoids

Abstract

We prove that a monoid M is a group if and only if, in the category of monoids, all points over M are strong. This sharpens and greatly simplifies a result of Montoli, Rodelo and Van der Linden [8] which characterises groups amongst monoids as the protomodular objects.

Reference

X. García-Martínez, *A new characterisation of groups amongst monoids*, Appl. Categ. Structures **25** (2017), no. 4, 659–661.

In their article [8], Montoli, Rodelo and Van der Linden introduce, amongst other things, the concept of a *protomodular object* in a finitely complete category $\mathcal{C}$ as an object $Y \in \mathcal{C}$ over which all points are *stably strong*. The aim of their definition is two-fold: first of all, to provide a categorical-algebraic characterisation of groups amongst monoids as the protomodular objects in the category Mon of monoids; and secondly, to establish an object-wise approach to certain important conditions occurring in categorial algebra such as protomodularity [2, 1] and the Mal'tsev axiom [5, 6].

We briefly recall some basic definitions; see [3, 7, 8] for more details. Let $\mathcal{C}$ be a finitely complete category, which we also take to be pointed for the sake of simplicity. In $\mathcal{C}$, a pair of arrows $(r\colon W \to X, s\colon Y \to X)$ is **jointly strongly epimorphic** when if $mr' = r$, $ms' = s$ for some given monomorphism $m\colon M \to X$ and arrows $r'\colon W \to M$, $s'\colon Y \to M$, then m is an isomorphism. In the case of monoids, this means that any element $x \in X$ can be written as a product $r(w_1)s(y_1)\cdots r(w_n)s(y_n)$ for some $w_j \in W$, $y_j \in Y$. This characterisation follows easily from the fact that (r, s) is a jointly strongly epimorphic pair in Mon if and only if the induced monoid morphism $W + Y \to X$ is a surjection—see, for instance, [1, Corollary A.5.4 combined with Example A.5.16]. Given an object Y in $\mathcal{C}$, a **point over** Y is a pair of morphisms $(f\colon X \to Y, s\colon Y \to X)$ such that $fs = 1_Y$. A point (f, s) is said to be **strong** when the pair $(\ker(f)\colon \mathsf{Ker}(f) \to X, s\colon Y \to X)$ is jointly strongly epimorphic. The point (f, s) is **stably strong** when all of its pullbacks are strong. More precisely, if $g\colon Z \to Y$ is any morphism, then the pullback $g^*(f)$ together with its splitting induced by s is a strong point.

Even though the concept of a protomodular object serves the intended purpose of characterising groups amongst monoids, the proof of this characterisation given in [8] is rather complicated, since it relies on another, more subtle, characterisation in terms of the so-called *Mal'tsev objects*. The present short note aims to improve the situation by giving a quick and direct proof of a more general result: a monoid is a group as soon as all points over it are strong.

Theorem 9.0.1. *A monoid M is a group if and only if, in* Mon*, all points over M are strong.*

Proof. It is shown in [4]—this is Proposition 2.2.4 combined with Lemma 2.1.6—that for any group M, all points over it are *homogenous*, which makes them (stably) strong. So in particular, if M is a group, then all points over M are strong. We prove the other implication.

Consider $m \in M$ and the induced split extension

$$0 \longrightarrow K \rightarrowtail \mathbb{N} + M \underset{(m\ 1_M)}{\overset{\iota_M}{\leftrightarrows}} M \longrightarrow 0,$$

where $m\colon \mathbb{N} \to M$ is the morphism which sends the generator 1 of $(\mathbb{N}, +, 0)$ to the element m of M. By the assumption that $((m\ 1_M), \iota_M)$ is a strong point,

$1 \in \mathbb{N}$ can be written as

$$1 = k_1 m_1 \cdots m_i k_{i+1} m_{i+1} \cdots k_n m_n$$

for some $k_j \in K$ and $m_j \in M$. Since 1 is not invertible in $\mathbb{N}$, it must appear in exactly one of the factors k_j in the product on the right, say in k_{i+1}. Then neither $k_1 m_1 \cdots m_i$ nor $m_{i+1} \cdots k_n m_n$ contains any non-zero elements of $\mathbb{N}$, so we have that in $\mathbb{N} + M$

$$1 = a'kb'$$

for some a', $b' \in M$ and $k \in K$. Since 1 appears in k we can write $k = a1b$ where a, $b \in M$. Necessarily then $e_M = a'a$ and $e_M = bb'$, because $1 = a'a \cdot 1 \cdot bb'$. Furthermore, since k is in the kernel of $(m\ 1_M)$, we also have that $e_M = amb$. So, clearly, a and b are invertible. As a consequence, m is invertible as well. We conclude that M is a group. □

Note that the above proof shows in particular why M is **gregarious** in the sense of [1], which means that for any m there exist a and b such that $e_M = amb$. However, the proof also shows that those a and b are invertible, and thus M is a group.

This result seems to indicate that in certain cases (like, for instance, in the category of monoids) it makes sense to weaken the definition of a protomodular object M—all points over M are stably strong—to the condition that those points are strong. This, and related considerations, will be the subject of future joint work with the authors of [8].

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Chapter 10

A note on split extensions of bialgebras

Abstract

We prove a universal characterization of Hopf algebras among cocommutative bialgebras over an algebraically closed field: a cocommutative bialgebra is a Hopf algebra precisely when every split extension over it admits a join decomposition. We also explain why this result cannot be extended to a non-cocommutative setting.

Reference

X. García-Martínez and T. Van der Linden, *A note on split extensions of bialgebras*, preprint `arXiv:1701.00665`, 2017.

10.1 Introduction

An elementary result in the theory of modules says that in any short exact sequence

$$0 \longrightarrow K \xrightarrow{k} X \underset{s}{\overset{f}{\rightleftarrows}} Y \longrightarrow 0 \qquad\qquad f \circ s = 1_Y$$

where the cokernel f admits a section s, the middle object X decomposes as a direct sum $X \cong K \oplus Y$. If, however, the given sequence is a short exact sequence of, say, groups or Lie algebras, then this is of course no longer true: then we can at most deduce that X is a semidirect product $K \rtimes Y$ of K and Y. In a fundamental way, this interpretation depends on, or even amounts to, the fact that X is generated by its subobjects $k(K)$ and $s(Y)$. One may argue that, in a non-additive setting, the join decomposition $X = k(K) \vee s(Y)$ in the lattice of subobjects of X is what replaces the direct sum decomposition, valid for split extensions of modules.

When the given split extension is a sequence of cocommutative bialgebras (over a commutative ring with unit $\mathbb{K}$), we may ask ourselves the question whether such a join decomposition of the middle object in the sequence always exists. Although kernels are not as nice as one could expect [2, 3], it is not difficult to see that *if Y is a Hopf algebra* then the answer is yes.

The main point of this note is that this happens *only then*, at least when $\mathbb{K}$ is an algebraically closed field. We shall prove, in other words, the following new universal characterization of cocommutative Hopf algebras among cocommutative bialgebras over $\mathbb{K}$:

> *All split extensions over a bialgebra Y admit a join decomposition if and only if Y is a Hopf algebra.*

This result is along the lines of, and is actually a variation on, a similar characterization of groups among monoids, recently obtained in [12, 7]. There the authors show that all split extensions (of monoids) over a monoid Y admit a join decomposition if and only if Y is a group.

In fact something stronger than the existence of a join decomposition may be proved in a more general context; this will be the subject of Section 10.2, where we explore some basic aspects of split extensions of cocommutative bialgebras. In particular, we show that over a Hopf algebra, all split extensions of cocommutative bialgebras admit a join decomposition (Corollary 10.2.5). In Section 10.3 we focus on the other implication and prove that among cocommutative bialgebras over an algebraically closed field, only Hopf algebras admit join decompositions of their split extensions (Theorem 10.3.5). In the final Section 10.4 we explain why the constraint that the bialgebras in this characterization are cocommutative is essential. As it turns out, in a non-cocommutative setting, even the very weakest universal join decomposition condition is too strong.

10.2 Split extensions over Hopf algebras

A **split extension** in a pointed category with finite limits $\mathcal{C}$ is a diagram

$$K \xrightarrow{k} X \underset{s}{\overset{f}{\rightleftarrows}} Y$$

where k is a kernel and s is a section of f. So $f \circ s = 1_Y$, but a priori we are not asking that f is a cokernel of k, so that (k, f) is a short exact sequence, and this is not automatically the case. We do always have that K and Y, considered as subobjects of X, have a trivial intersection. Indeed, using that k is the pullback of $0 \to Y$ along f, it is easy to check that the pullback of k and s is zero.

In this general context, a join of two subobjects may not always exist, but the concept introduced in the next definition expresses what we want, and agrees with the condition that $X = k(K) \vee s(Y)$ whenever that expression makes sense—as it does in any regular category with binary coproducts, for instance [4].

Definition 10.2.1. A pair of arrows (k, s) with the same codomain X is **jointly extremally epimorphic** when the arrows k and s cannot both factor through one and the same proper subobject of X: whenever we have a diagram

$$\begin{array}{ccccc} & & M & & \\ & \nearrow & \downarrow m & \nwarrow & \\ K & \xrightarrow[k]{} & X & \xleftarrow[s]{} & Y \end{array}$$

where m is a monomorphism, necessarily m is an isomorphism. We say that a split extension as above is **strong** when (k, s) is a jointly extremally epimorphic pair; the couple (f, s) is then called a **strong point**. When we say that a split extension **admits a join decomposition**, we mean that it is strong.

The given split extension is said to be **stably strong** (the couple (f, s) is a **stably strong point**) when all of its pullbacks (along any morphism $g \colon W \to Y$) are strong. Following [12], we say that an object Y is **protomodular** when all split extensions over Y are stably strong.

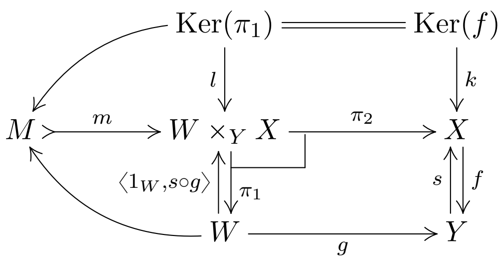
Remark 10.2.2. It is easily seen [12] that the split epimorphism f in a strong point (f, s) is always the cokernel of its kernel k. This means, in particular, that all split extensions over a protomodular object Y, as well as all of their pullbacks, are (split) short exact sequences which admit a join decomposition.

Remark 10.2.3. When all objects in $\mathcal{C}$ are protomodular, $\mathcal{C}$ is a **protomodular category** in the sense of [5]. Next to Barr exactness, this is one of the key ingredients in the definition of a semi-abelian category [9], and crucial for results such as the 3×3 *Lemma*, the *Snake Lemma*, the *Short Five Lemma* [6, 4], or the existence of a Quillen model category structure for homotopy of simplicial objects [15]. Typical examples are the categories of groups, Lie algebras, crossed modules, loops, associative algebras, etc. As recently shown in [8, 10], also the category of cocommutative Hopf algebras over a field of characteristic zero is semi-abelian.

Given a category with finite products $\mathcal{C}$, we write $\mathsf{Mon}(\mathcal{C})$ for the category of internal monoids, and $\mathsf{Gp}(\mathcal{C})$ for the category of internal groups in $\mathcal{C}$. For a commutative ring with unit $\mathbb{K}$, we let $\mathsf{CoAlg}_{\mathbb{K},coc}$ denote the category of cocommutative coalgebras over $\mathbb{K}$. It is well known [14] that there is an equivalence between the category $\mathsf{BiAlg}_{\mathbb{K},coc}$ of cocommutative bialgebras over $\mathbb{K}$ and $\mathsf{Mon}(\mathsf{CoAlg}_{\mathbb{K},coc})$, which restricts to an equivalence between the category $\mathsf{Hopf}_{\mathbb{K},coc}$ of cocommutative Hopf algebras over $\mathbb{K}$ and $\mathsf{Gp}(\mathsf{CoAlg}_{\mathbb{K},coc})$. This is easily seen using that in $\mathsf{CoAlg}_{\mathbb{K},coc}$ the product $X \times Y$ is $X \otimes Y$ and 1 is $\mathbb{K}$.

Theorem 10.2.4. *Let $\mathcal{C}$ be a category with finite limits. If $Y \in \mathsf{Gp}(\mathcal{C})$ then all split extensions in $\mathsf{Mon}(\mathcal{C})$ over Y are stably strong. In other words, any internal group in $\mathcal{C}$ is a protomodular object in $\mathsf{Mon}(\mathcal{C})$.*

Proof. Consider in $\mathsf{Mon}(\mathcal{C})$ the commutative diagram



where the bottom right square is a pullback, m is a monomorphism, and Y is an internal group. We shall see that m is an isomorphism. Since only limits are considered, the whole commutative diagram is sent into a category of presheaves of sets by the Yoneda embedding, in such a way that the internal groups and internal monoids in it are mapped to ordinary groups and monoids, respectively. Since the Yoneda embedding reflects isomorphisms, it now suffices to give a proof in Set. There, it is easy to see that m is an isomorphism, since every element (w, x) of $W \times_Y X$ can be written as $(1, x \cdot s(g(w)^{-1})) \cdot (w, sg(w))$, where clearly the first element belongs to the kernel of π_1 and the second one comes from W. □

Corollary 10.2.5. *Cocommutative Hopf algebras are protomodular in* $\mathsf{BiAlg}_{\mathbb{K},coc}$. □

It follows that, over a Hopf algebra, split extensions of bialgebras are well-behaved; not only are they short exact sequences, but it is also not hard to see that the *Split Short Five Lemma* holds for them, so that equivalences classes of split extensions may be considered as in ordinary group cohomology.

10.3 A universal characterization of cocommutative Hopf algebras

The converse is less straightforward. In the case of groups and monoids ($\mathcal{C} = \mathsf{Set}$ in Theorem 10.2.4), it was shown in [12] (resp. in [7]) that all points in Mon over Y are stably strong (resp. strong) if and only if Y is a group. However, those proofs involve coproducts, and so a Yoneda embedding argument as in Theorem 10.2.4 would not work.

We now let $\mathbb{K}$ be an algebraically closed field. We consider the adjoint pair

$$\mathsf{BiAlg}_{\mathbb{K},coc} \underset{\mathbb{K}[-]}{\overset{G}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} \mathsf{Mon}$$

where the left adjoint $\mathbb{K}[-]$ is the monoid algebra functor and the right adjoint G sends a bialgebra B (with comultiplication Δ_B and counit ε_B) to its monoid of grouplike elements $G(B) = \{x \in B \mid \Delta_B(x) = x \otimes x \text{ and } \varepsilon_B(x) = 1\}$.

Lemma 10.3.1. $\mathbb{K}[-]$ *preserves monomorphisms.*

Proof. The functor $\mathbb{K}[-]$ sends any monoid monomorphism to a bialgebra morphism of which the underlying vector space map is an injection. □

Our aim is to prove that G preserves protomodular objects: then for any protomodular bialgebra B, the monoid of grouplike elements $G(B)$ is a group, so that B is a Hopf algebra by [14, 8.0.1.c and 9.2.5].

Proposition 10.3.2. *For any monoid M we have $G(\mathbb{K}[M]) \cong M$. For any bialgebra B, the counit $\epsilon_B \colon \mathbb{K}[G(B)] \to B$ of the adjunction at B is a split monomorphism with retraction $\pi_B \colon B \to \mathbb{K}[G(B)]$, determined in a way which is functorial in B.*

Proof. The first statement follows immediately from the definition of $\mathbb{K}[M]$, while the second depends on [14, 8.0.1.c and 8.1.2]. □

Since protomodular objects are closed under retracts [12], it follows that if B is a protomodular bialgebra, then so is $\mathbb{K}[G(B)]$.

Proposition 10.3.3. *The functor G preserves jointly extremally epimorphic pairs.*

Proof. Let (k, s) be a jointly extremally epimorphic pair in $\mathsf{BiAlg}_{\mathbb{K},coc}$. Then the commutativity of the diagram

$$\begin{array}{ccccc} \mathbb{K}[G(K)] & \xrightarrow{\mathbb{K}[G(k)]} & \mathbb{K}[G(X)] & \xleftarrow{\mathbb{K}[G(s)]} & \mathbb{K}[G(Y)] \\ {\scriptstyle \pi_K}\uparrow & & {\scriptstyle \pi_X}\uparrow & & \uparrow{\scriptstyle \pi_Y} \\ K & \xrightarrow[k]{} & X & \xleftarrow[s]{} & Y \end{array}$$

obtained via Proposition 10.3.2 and the fact that the upward pointing arrows are split epimorphisms imply that the pair $(\mathbb{K}[G(k)], \mathbb{K}[G(s)])$ is jointly extremally epimorphic. Now suppose that m is a monomorphism making the diagram on the left

$$\begin{array}{ccccc} & & M & & \\ & \nearrow & \big\downarrow{\scriptstyle m} & \nwarrow & \\ G(K) & \xrightarrow[G(k)]{} & G(X) & \xleftarrow[G(s)]{} & G(Y) \end{array} \qquad \begin{array}{ccccc} & & \mathbb{K}[M] & & \\ & \nearrow & \big\downarrow{\scriptstyle \mathbb{K}[m]} & \nwarrow & \\ \mathbb{K}[G(K)] & \xrightarrow[\mathbb{K}[G(k)]]{} & \mathbb{K}[G(X)] & \xleftarrow[\mathbb{K}[G(s)]]{} & \mathbb{K}[G(Y)] \end{array}$$

commute. Applying $\mathbb{K}[-]$ we obtain the diagram on the right, in which $\mathbb{K}[m]$ is a monomorphism by Lemma 10.3.1. Since, by the above, the bottom pair is jointly extremally epimorphic, we see that $\mathbb{K}[m]$ is an isomorphism. But then also $m = G(\mathbb{K}[m])$ is an isomorphism, which proves our claim that $(G(k), G(s))$ is a jointly extremally epimorphic pair. □

Proposition 10.3.4. *If all split extensions over a bialgebra Y are strong, then all split extensions over $G(Y)$ are strong. In particular, G preserves protomodular objects.*

Proof. Consider a split extension

$$K \xrightarrow{k} X \underset{s}{\overset{f}{\rightleftarrows}} G(Y)$$

over $G(Y)$. We apply the functor $\mathbb{K}[-]$, then take the kernel of $\mathbb{K}[f]$ to obtain the split extension of bialgebras

$$L \xrightarrow{l} \mathbb{K}[X] \underset{\mathbb{K}[s]}{\overset{\mathbb{K}[f]}{\rightleftarrows}} \mathbb{K}[G(Y)].$$

From Proposition 10.3.2 it follows that all split extensions over $\mathbb{K}[G(Y)]$ are strong. Hence $(l, \mathbb{K}[s])$ is a jointly extremally epimorphic pair. Applying the functor G, we regain the original split extension, since G is a right adjoint, thus preserves kernels; but G also preserves jointly extremally epimorphic pairs by Proposition 10.3.3, so that the pair (k, s) is jointly extremally epimorphic. As a consequence, all split extensions over the monoid $G(Y)$ are strong, and $G(Y)$ is protomodular [7]. □

Theorem 10.3.5. *If $\mathbb{K}$ is an algebraically closed field and Y is a cocommutative bialgebra over $\mathbb{K}$, then the following conditions are equivalent:*

(i) Y is a Hopf algebra;

(ii) in $\mathsf{BiAlg}_{\mathbb{K},coc}$, all split extensions over Y admit a join decomposition;

(iii) Y is a protomodular object in $\mathsf{BiAlg}_{\mathbb{K},coc}$.

Proof. (i) implies (iii) is Theorem 10.2.4, and (ii) is obviously weaker than (iii). For the proof that (ii) implies (i), suppose that all split extensions over Y admit a join decomposition. Then Proposition 10.3.4 implies that in Mon all split extensions over $G(Y)$ are strong. Hence $G(Y)$ is a group by the result in [7], which makes Y a Hopf algebra by [14, 8.0.1.c and 9.2.5]. □

Remark 10.3.6. This implies that the category $\mathsf{BiAlg}_{\mathbb{K},coc}$ cannot be protomodular: otherwise all bialgebras would be Hopf algebras. In particular, the *Split Short Five Lemma* is not generally valid for bialgebras.

10.4 On cocommutativity

In this final section we study what happens beyond the cocommutative setting. Here $\mathbb{K}$ is a field.

All objects in the category of cocommutative $\mathbb{K}$-bialgebras satisfy a certain weak join decomposition property: being a category of internal monoids (in $\mathsf{CoAlg}_{\mathbb{K},coc}$), the category $\mathsf{BiAlg}_{\mathbb{K},coc}$ is **unital** in the sense of [4]. Given an object Y, it is said to be a **unital object** [12] when every split extension of the type

$$X \underset{\langle 1_X,0\rangle}{\overset{\pi_X}{\leftrightarrows}} X \times Y \underset{\langle 0,1_Y\rangle}{\overset{\pi_Y}{\rightleftarrows}} Y$$

is strong. Notice how this condition is symmetric in X and Y. So protomodular objects are always unital of course, but in fact this condition is weak enough to be satisfied by all cocommutative bialgebras over $\mathbb{K}$.

Let us now leave the cocommutative setting and ask ourselves what it means for an object Y in $\mathsf{BiAlg}_{\mathbb{K}}$ to be unital—a very weak thing to ask, compared with the condition that all split extensions over Y are (stably) strong.

Proposition 10.4.1. *If Y is a unital object of $\mathsf{BiAlg}_{\mathbb{K}}$, then for every object X we have an isomorphism $X \times Y \cong X \otimes Y$.*

Proof. Given any bialgebra X we may consider the diagram

$$X \xleftarrow[\cong]{\rho_X} X \otimes \mathbb{K} \underset{1_X\otimes\eta_Y}{\overset{1_X\otimes\varepsilon_Y}{\leftrightarrows}} X \otimes Y \underset{\eta_X\otimes 1_Y}{\overset{\varepsilon_X\otimes 1_Y}{\rightleftarrows}} \mathbb{K} \otimes Y \xrightarrow[\cong]{\lambda_Y} Y.$$

We are first going to prove that the comparison morphism

$$m = \langle \rho_X \circ (1_X \otimes \varepsilon_Y), \lambda_Y \circ (\varepsilon_X \otimes 1_Y)\rangle\colon X \otimes Y \to X \times Y$$

is a monomorphism.

Note that it is almost never an injection; for instance, taking $X = Y$ to be a tensor algebra $T(V)$ (with counit $\varepsilon_{T(V)}(v) = 0$ for $v \in V$) yields easy counterexamples. However, in the category $\mathsf{BiAlg}_{\mathbb{K}}$, monomorphisms need not be injective [13, 1].

Let $h\colon Z \to X \otimes Y$ be a morphism of bialgebras. We write

$$f = \rho_X \circ (1_X \otimes \varepsilon_Y) \circ h\colon Z \to X \qquad \text{and} \qquad g = \lambda_Y \circ (\varepsilon_X \otimes 1_Y) \circ h\colon Z \to Y.$$

It suffices to prove that $h = (f \otimes g) \circ \Delta_Z$ *as vector space maps* for our claim to hold. Indeed, if h and h' induce the same f and g, then the given equality of vector space maps proves that $h = h'$.

Since h is a coalgebra map, we have that $\Delta_{X \otimes Y} \circ h = (h \otimes h) \circ \Delta_Z$. Writing $\tau_{X,Y} \colon X \otimes Y \to Y \otimes X$ for the twist map, we calculate:

$$\begin{aligned}
&(f \otimes g) \circ \Delta_Z \\
&= (\rho_X \otimes \lambda_Y) \circ (1_X \otimes \varepsilon_Y \otimes \varepsilon_X \otimes 1_Y) \circ (h \otimes h) \circ \Delta_Z \\
&= (\rho_X \otimes \lambda_Y) \circ (1_X \otimes \varepsilon_Y \otimes \varepsilon_X \otimes 1_Y) \circ \Delta_{X \otimes Y} \circ h \\
&= (\rho_X \otimes \lambda_Y) \circ (1_X \otimes \varepsilon_Y \otimes \varepsilon_X \otimes 1_Y) \circ (1_X \otimes \tau_{X,Y} \otimes 1_Y) \circ (\Delta_X \otimes \Delta_Y) \circ h \\
&= (\rho_X \otimes \lambda_Y) \circ (1_X \otimes \varepsilon_X \otimes \varepsilon_Y \otimes 1_Y) \circ (\Delta_X \otimes \Delta_Y) \circ h \\
&= (\rho_X \otimes \lambda_Y) \circ (\rho_X^{-1} \otimes \lambda_Y^{-1}) \circ h = h.
\end{aligned}$$

It follows that m is a monomorphism. Moreover, m makes the diagram

$$\begin{array}{ccccc}
 & & X \otimes Y & & \\
 & \nearrow^{(1_X \otimes \eta_Y) \circ \rho_X^{-1}} & \downarrow m & \nwarrow^{(\eta_X \otimes 1_Y) \circ \lambda_Y^{-1}} & \\
X & \xrightarrow[\langle 1_X, 0 \rangle]{} & X \times Y & \xleftarrow[\langle 0, 1_Y \rangle]{} & Y
\end{array}$$

commute. The assumption that Y is unital tells us that m is an isomorphism. □

This immediately implies that any unital object Y in $\mathsf{BiAlg}_{\mathbb{K}}$ has to be cocommutative, since $\Delta_Y \colon Y \to Y \otimes Y$ is the morphism of bialgebras $\langle 1_Y, 1_Y \rangle \colon Y \to Y \times Y$. In particular, the category $\mathsf{BiAlg}_{\mathbb{K}}$ is not unital, so it cannot be protomodular, and not even Mal'tsev [4].

However, the situation is actually much worse, since it almost never happens that $X \otimes Y$ is the product of X in Y in the category of all $\mathbb{K}$-bialgebras—not even when both X and Y are cocommutative. In fact, $\mathbb{K}$ itself cannot be a protomodular object in $\mathsf{BiAlg}_{\mathbb{K}}$, since this would imply that all objects of $\mathsf{BiAlg}_{\mathbb{K}}$ are unital [12]. As we have just seen, this is manifestly false.

The same holds for the category $\mathsf{Hopf}_{\mathbb{K}}$ of Hopf algebras over $\mathbb{K}$. At first this may seem to contradict results in [11] on split extensions of Hopf algebras. We must keep in mind, though, that for a Hopf algebra H, the map $\langle 1_H, 0 \rangle$ in the diagram

$$H \underset{\langle 1_H, 0 \rangle}{\overset{\pi_1}{\leftrightarrows}} H \times H \underset{\langle 0, 1_H \rangle}{\overset{\pi_2}{\rightleftarrows}} H$$

is the kernel of π_2, but π_2 need not be its cokernel, unless H is cocommutative. Hence this diagram does not represent a short exact sequence, and so neither Theorem 4.1 nor Theorem 4.2 in [11] saying that $H \times H \cong H \otimes H$ applies.

We conclude that it makes no sense to study protomodular objects in $\mathsf{BiAlg}_{\mathbb{K}}$ or in $\mathsf{Hopf}_{\mathbb{K}}$, and we thus restrict our attention to the cocommutative case.

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Chapter 11

A characterisation of Lie algebras amongst alternating algebras

Abstract

Let $\mathbb{K}$ be an infinite field. We prove that if a variety of alternating $\mathbb{K}$-algebras—not necessarily associative, where $xx = 0$ is a law—is locally algebraically cartesian closed, then it must be a variety of Lie algebras over $\mathbb{K}$. In particular, $\mathsf{Lie}_{\mathbb{K}}$ is the largest such. Thus, for a given variety of alternating $\mathbb{K}$-algebras, the Jacobi identity becomes equivalent to a categorical condition: it is a law in $\mathcal{V}$ if and only if $\mathcal{V}$ is a subvariety of a locally algebraically cartesian closed variety of alternating $\mathbb{K}$-algebras. This is based on a result saying that an algebraically coherent variety of alternating $\mathbb{K}$-algebras is either a variety of Lie algebras or a variety of antiassociative algebras over $\mathbb{K}$.

Reference

X. García-Martínez and T. Van der Linden, *A characterisation of Lie algebras amongst alternating algebras*, preprint `arXiv:1701.05493`, 2017.

11.1 Introduction

The aim of this article is to prove that, if a variety of alternating algebras—not necessarily associative, where $xx = 0$ is a law—over an infinite field admits *algebraic exponents* in the sense of James Gray's Ph.D. thesis [15], so when it is *locally algebraically cartesian closed* (or (LACC) for short, see [17, 7]), then it must necessarily be a variety of Lie algebras. Since, as shown in [16], the category $\mathsf{Lie}_{\mathbb{K}}$ of Lie algebras over a commutative unitary ring $\mathbb{K}$ is always (LACC), this condition may be used to characterise Lie algebras amongst alternating algebras.

The only other non-abelian "natural" examples of locally algebraically cartesian closed semi-abelian [20] categories we currently know of happen to be categories of group objects in a cartesian closed category [17], namely

1. the category Gp of groups itself;

2. the category XMod of crossed modules, which are the group objects in the category Cat of small categories [22, 24]; and

3. the category $\mathsf{Hopf}_{\mathbb{K},\mathsf{coc}}$ of cocommutative Hopf algebras over a field $\mathbb{K}$ of characteristic zero [14, 10], the group objects in the category $\mathsf{CoAlg}_{\mathbb{K},\mathsf{coc}}$ of cocommutative coalgebras over $\mathbb{K}$.

At first with our project we hoped to remedy this situation by finding further examples of (LACC) categories of (not necessarily associative) algebras. However, all of our attempts at constructing such new examples failed. Quite unexpectedly, in the end we managed to prove that, at least when the field $\mathbb{K}$ is infinite, amongst those algebras which are alternating, *there are no other examples*: the condition (LACC) implies that the Jacobi identity holds. Thus, in the context of alternating algebras, the Jacobi identity is characterised in terms of a purely categorical condition. This is the subject of Section 11.2.

We do not know what happens when the algebras considered are not alternating. The category of Leibniz algebras is not (LACC), so at least one of the implications in our characterisation fails in that case. We make a few additional observations in Section 11.3, and hope to study this question in future work.

11.1.1 Cartesian closedness

Algebraic exponentiation is a categorical-algebraic version of the concept of *exponentiation* familiar from set theory, linear algebra, topology, etc. In its most basic form, exponentiation amounts to the task of equipping the set

$\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of morphisms from X to Y with a suitable structure making it an object Y^X in the category $\mathcal{C}$ at hand.

Depending on the given category $\mathcal{C}$, this may or may not be always possible. A category with binary products $\mathcal{C}$ is said to be **cartesian closed** when for every object X the functor $X \times (-)\colon \mathcal{C} \to \mathcal{C}$ admits a right adjoint $(-)^X\colon \mathcal{C} \to \mathcal{C}$, so that for all Y and Z in $\mathcal{C}$, the set $\mathrm{Hom}_{\mathcal{C}}(X \times Z, Y)$ is isomorphic to $\mathrm{Hom}_{\mathcal{C}}(Z, Y^X)$. In particular then, an object Y^X exists for all X and Y; see [24] for further details.

The category Set of sets is cartesian closed, with Y^X the set $\mathrm{Hom}_{\mathsf{Set}}(X, Y)$ of functions from X to Y. Also the category Cat of small categories is cartesian closed. The category Y^X has functors $X \to Y$ as objects, and natural transformations between them as morphisms. For any commutative ring $\mathbb{K}$, the category $\mathsf{CoAlg}_{\mathbb{K},\mathsf{coc}}$ of cocommutative coalgebras over $\mathbb{K}$ is cartesian closed by a result in [1].

11.1.2 Closedness in general

The categories occurring in algebra are seldom cartesian closed. The concept of closedness has thus been extended in several different directions. One option is to replace the cartesian product by some other product, such as for instance the tensor product $\otimes_{\mathbb{K}}$ when $\mathcal{C}$ is the category $\mathsf{Vect}_{\mathbb{K}}$ of vector spaces over a field $\mathbb{K}$. In that case the result is the well-known tensor/hom adjunction, where the object Y^X in the isomorphism $\mathrm{Hom}_{\mathbb{K}}(X \otimes_{\mathbb{K}} Z, Y) \cong \mathrm{Hom}_{\mathbb{K}}(Z, Y^X)$ is the set of $\mathbb{K}$-linear maps $\mathrm{Hom}_{\mathbb{K}}(X, Y)$ with the pointwise $\mathbb{K}$-vector space structure.

11.1.3 An alternative approach

Another option, fruitful in non-abelian algebra, is to keep the cartesianness aspect of the condition, but to make it algebraic in an entirely different way [15, 17, 7]. To do this, we first need to understand what is *local* cartesian closedness by reformulating the condition in terms of slice categories. Here we follow Section A1.5 of [21].

11.1.4 Local cartesian closedness

Let $\mathcal{C}$ be any category. Given an object B of $\mathcal{C}$, we write $(\mathcal{C} \downarrow B)$ for the **slice category** or **category of objects over** B in which an object x is an arrow

$x\colon X \to B$ in $\mathcal{C}$, and a morphism $f\colon x \to y$ is a commutative triangle

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle x}\searrow \quad \swarrow{\scriptstyle y} & \\ & B & \end{array}$$

in $\mathcal{C}$, so that $y \circ f = x$.

Assuming now that $\mathcal{C}$ is finitely complete, given a morphism $a\colon A \to B$, we write

$$a^*\colon (\mathcal{C} \downarrow B) \to (\mathcal{C} \downarrow A)$$

for the **change-of-base functor** which takes an arrow $x\colon X \to B$ in $\mathcal{C}$ and sends it to its pullback $a^*(x)$ as in the diagram

$$\begin{array}{ccc} A \times_B X & \longrightarrow & X \\ {\scriptstyle a^*(x)}\downarrow & \lrcorner & \downarrow{\scriptstyle x} \\ A & \xrightarrow[a]{} & B. \end{array}$$

If B is the terminal object 1 of $\mathcal{C}$ then $(\mathcal{C} \downarrow B) = (\mathcal{C} \downarrow 1) \cong \mathcal{C}$. Any object A of $\mathcal{C}$ now induces a unique morphism $a = !_A\colon A \to 1$, and the functor $!_A^*\colon \mathcal{C} \to (\mathcal{C} \downarrow A)$ sends an object Y to the product $A \times Y$ (considered together with its projection to A). It is easily seen that the category $\mathcal{C}$ is cartesian closed if and only if for every X in $\mathcal{C}$, the functor $!_X^*$ admits a right adjoint.

A category with finite limits $\mathcal{C}$ is said to be **locally cartesian closed** or **(LCC)** when for *every* morphism $a\colon A \to B$ in $\mathcal{C}$ the change-of-base functor a^* has a right adjoint. Equivalently, all slice categories $(\mathcal{C} \downarrow B)$ are cartesian closed—so that $\mathcal{C}$ is cartesian closed, *locally over* B, for all B in $\mathcal{C}$. This condition is stronger than cartesian closedness (the case $B = 1$); examples include any Grothendieck topos, in particular the category of sets, while for instance [12] the category Cat is not (LCC), even though it is cartesian closed.

11.1.5 Categories of points

We may now modify the concept of (local) cartesian closedness in such a way that it applies to algebraic categories. The idea is that, where *slice categories* are useful in non-algebraic settings, in algebraic categories a similar role may be played by *categories of points*.

Let $\mathcal{C}$ be any category. Given an object B of $\mathcal{C}$, we write $\mathsf{Pt}_B(\mathcal{C})$ for the **category of points over** B in which an object (x, s) is a split epimorphism $x\colon X \to B$ in $\mathcal{C}$, together with a chosen section $s\colon B \to X$, so that $x \circ s = 1_B$. Given two points $(x\colon X \to B, s\colon B \to X)$ and $(y\colon Y \to B, t\colon B \to Y)$ over B, a morphism between them is an arrow $f\colon X \to Y$ in $\mathcal{C}$ satisfying $y \circ f = x$ and $f \circ s = t$.

Change of base is done as for slice categories: since sections are preserved, given any morphism $a\colon A \to B$ in a finitely complete category $\mathcal{C}$, we obtain a functor

$$a^*\colon \mathsf{Pt}_B(\mathcal{C}) \to \mathsf{Pt}_A(\mathcal{C}).$$

11.1.6 Protomodular and semi-abelian categories

A finitely complete category $\mathcal{C}$ is said to be **Bourn protomodular** [3, 5, 2] when each of the change-of-base functors $a^*\colon \mathsf{Pt}_B(\mathcal{C}) \to \mathsf{Pt}_A(\mathcal{C})$ reflect isomorphisms. If $\mathcal{C}$ is a pointed category, then this condition may be reduced to the special case where A is the zero object and $a = \text{¡}_B\colon 0 \to B$ is the unique morphism. The pullback functor $\text{¡}_B^*\colon \mathsf{Pt}_B(\mathcal{C}) \to \mathcal{C}$ then sends a split epimorphism to its kernel. Hence, protomodularity means that the **Split Short Five Lemma** holds: suppose that in the commutative diagram

$$\begin{array}{ccccc} K & \xrightarrow{k} & X & \overset{x}{\underset{s}{\rightleftarrows}} & B \\ {\scriptstyle g}\downarrow & & {\scriptstyle f}\downarrow & & \| \\ L & \xrightarrow[l]{} & Y & \overset{y}{\underset{t}{\rightleftarrows}} & B, \end{array}$$

the morphism k is the kernel of x and l is the kernel of y, while f is a morphism of points $(x, s) \to (y, t)$; if now g is an isomorphism, then f is also an isomorphism.

A pointed protomodular category which is Barr exact and has finite coproducts is called a **semi-abelian** category [20]. This concept unifies earlier attempts (including, for instance, [19, 13, 28]) at providing a categorical framework that extends the context of abelian categories to encompass non-additive categories of algebraic structures such as groups, Lie algebras, loops, rings, etc. In this setting, the basic lemmas of homological algebra—the 3×3 *Lemma*, the *Short Five Lemma*, the *Snake Lemma*—hold [5, 2], and may be used to study, say, (co)homology, radical theory, or commutator theory for those non-additive structures.

In a semi-abelian category, any point (x, s) with its induced kernel k as above gives rise to a split extension, since x is also the cokernel of k, so that (k, x) is a short exact sequence. By the results in [8], split extensions are equivalent to so-called *internal actions* by means of a semi-direct product construction. Through this equivalence, there is a unique internal action $\xi\colon B\flat K \to K$ such that $X \cong K \rtimes_\xi B$. Without going into further details, let us just mention here that the object $B\flat K$ is the kernel of the morphism $(1_B\ 0)\colon B + K \to B$, that the functor $B\flat(-)\colon \mathcal{C} \to \mathcal{C}$ is part of a monad, and that an internal B-action is an algebra for this monad. The category $\mathsf{Pt}_B(\mathcal{C})$ is monadic over $\mathcal{C}$, and its equivalence with the category of $B\flat(-)$-algebras bears witness of this fact.

11.1.7 Examples

All *Higgins varieties of* Ω*-groups* [18] are semi-abelian, which means that any pointed variety of universal algebras whose theory contains a group operation is an example. In particular, we find categories of all kinds of (not necessarily associative) algebras over a ring as examples, next to the categories of groups, crossed modules, and groups of a certain nilpotency or solvability class. Other examples include the categories of Heyting semilattices, loops, compact Hausdorff groups and the dual of the category of pointed sets [20, 2].

11.1.8 Algebraic cartesian closedness and the condition (LACC)

A category with finite limits $\mathcal{C}$ is said to be **locally algebraically cartesian closed** or **(LACC)** when for every morphism $a\colon A \to B$ in $\mathcal{C}$, the change-of-base functor $a^*\colon \mathsf{Pt}_B(\mathcal{C}) \to \mathsf{Pt}_A(\mathcal{C})$ has a right adjoint [15]. This condition is much stronger than **algebraic cartesian closedness** or **(ACC)** which is the case $B = 1$.

When a semi-abelian category is (locally) algebraically cartesian closed, this has some interesting consequences [17, 7, 10]. For one thing, (ACC) is equivalent to the condition that every monomorphism in $\mathcal{C}$ admits a centraliser. The property (LACC) implies categorical-algebraic conditions such as *peri-abelianness* [6], *strong protomodularity* [4], the *Smith is Huq* condition [25], *normality of Higgins commutators* [11], and *algebraic coherence*. We come back to the latter condition (which implies all the others mentioned) in detail, in Subsection 11.1.9 below.

The condition (ACC) is relatively weak, and has all *Orzech categories of interest* [28] for examples. In comparison, (LACC) is very strong: as mentioned above, we have groups, Lie algebras, crossed modules, and cocommutative Hopf algebras over a field of characteristic zero as "natural" semi-abelian examples, next to all abelian categories. An example of a slightly different kind—because it is non-pointed—is any category of groupoids with a fixed object of objects [7].

In what follows, we shall need the following characterisation of (LACC), valid in semi-abelian varieties of universal algebras. Instead of checking that all change-of-base functors $a^*\colon \mathsf{Pt}_B(\mathcal{V}) \to \mathsf{Pt}_A(\mathcal{V})$ have a right adjoint, it suffices to check that *some* change-of-base functors preserve binary sums.

Theorem 11.1.1. *For a semi-abelian variety of universal algebras $\mathcal{V}$, the following are equivalent:*

(i) $\mathcal{V}$ is locally algebraically cartesian closed;

(ii) for all B in $\mathcal{V}$, the pullback functor $¡_B^*\colon \mathsf{Pt}_B(\mathcal{V}) \to \mathcal{V}$ preserves all colimits;

(iii) for all B in $\mathcal{V}$, the functor $¡_B^*$ preserves binary sums;

(iv) the canonical comparison $\lfloor B\flat\iota_X\ B\flat\iota_Y \rceil\colon B\flat X + B\flat Y \to B\flat(X+Y)$ is an isomorphism for all B, X and Y in $\mathcal{V}$.

Proof. This combines Theorem 2.9, Theorem 5.1 and Proposition 6.1 in [17]. □

Via the equivalence between split extensions and internal actions, condition (ii) means that the forgetful functor from the category of B-actions in $\mathcal{V}$ to $\mathcal{V}$ preserves all colimits. Hence in this varietal context, (LACC) amounts to the property that colimits in the category of internal B-actions in $\mathcal{V}$ are independent of the acting object B, and computed in the base category $\mathcal{V}$.

11.1.9 Algebraic coherence

The concept of an **algebraically coherent** category was introduced in [10] with the aim in mind of having a condition with strong categorical-algebraic consequences such as the ones mentioned above for (LACC), while at the same time keeping all *Orzech categories of interest* as examples. It is to *coherence* in the sense of topos theory [21, Section A1.4] what *algebraic cartesian closedness* is to *cartesian closedness*: a condition involving slice categories has been replaced by a condition in terms of categories of points.

The formal definition is that all change-of-base functors $a^*\colon \mathsf{Pt}_B(\mathcal{C}) \to \mathsf{Pt}_A(\mathcal{C})$ preserve jointly strongly epimorphic pairs of arrows. This is clearly weaker than asking that the a^* preserve all colimits. We shall only need the following characterisation, which is essentially Theorem 3.18 in [10]: algebraic coherence is equivalent to the condition that for all B, X and Y, the canonical comparison

$$(B\flat\iota_X \; B\flat\iota_Y)\colon B\flat X + B\flat Y \to B\flat(X+Y)$$

from Theorem 11.1.1 is a regular epimorphism.

Algebraic coherence has somewhat better stability properties than (LACC). For instance, any subvariety of a semi-abelian algebraically coherent variety is still algebraically coherent. We shall come back to this in the next section.

Some examples of semi-abelian varieties which are *not* algebraically coherent are the varieties of loops, Heyting semilattices, and non-associative algebras (the category $\mathsf{Alg}_{\mathbb{K}}$ defined below).

11.2 Main result

The aim of this section is to prove Theorem 11.2.9, which says that any (LACC) variety of alternating algebras over an infinite field $\mathbb{K}$ is a category of Lie algebras over $\mathbb{K}$. On the way we fully characterise algebraically coherent varieties of alternating algebras (Theorem 11.2.7). This is an application of a more general result telling us that a variety of $\mathbb{K}$-algebras is algebraically coherent if and only if it is an *Orzech category of interest* (Theorem 11.2.5).

11.2.1 Categories of algebras and their subvarieties

Let $\mathbb{K}$ be a field. A **(non-associative) algebra** A **over** $\mathbb{K}$ is a $\mathbb{K}$-vector space equipped with a bilinear operation $[-,-]\colon A \times A \to A$, so a linear map $A \otimes A \to A$. We use the notations $[x,y] = x \cdot y = xy$ depending on the situation at hand, always keeping in mind that the multiplication need not be associative. We write $\mathsf{Alg}_{\mathbb{K}}$ for the category of algebras over $\mathbb{K}$ with product-preserving linear maps between them. It is a semi-abelian category which is not algebraically coherent. A **subvariety** of $\mathsf{Alg}_{\mathbb{K}}$ is any equationally defined class of algebras, considered as a full subcategory $\mathcal{V}$ of $\mathsf{Alg}_{\mathbb{K}}$.

The category of **associative algebras** over $\mathbb{K}$ is the subvariety of $\mathsf{Alg}_{\mathbb{K}}$ satisfying $x(yz) = (xy)z$.

The category $\mathsf{Alt}_{\mathbb{K}}$ of **alternating algebras** over $\mathbb{K}$ is the subvariety of $\mathsf{Alg}_{\mathbb{K}}$ satisfying $xx = 0$. It is easily seen that those algebras are **anticommutative**, which means that $xy = -yx$ holds. If the characteristic of the field $\mathbb{K}$ is different from 2, then the two classes coincide.

The category $\mathsf{AAAlg}_{\mathbb{K}}$ of **antiassociative algebras** over $\mathbb{K}$ is the subvariety of $\mathsf{Alg}_{\mathbb{K}}$ satisfying $x(yz) = -(xy)z$. We write $\mathsf{AAAAlg}_{\mathbb{K}}$ for the category of alternating antiassociative algebras over $\mathbb{K}$.

The category $\mathsf{Lie}_{\mathbb{K}}$ of **Lie algebras** over $\mathbb{K}$ is the subvariety of alternating algebras satisfying the **Jacobi identity** $x(yz) + z(xy) + y(zx) = 0$.

An algebra is **abelian** when it satisfies $xy = 0$. The subvariety of $\mathsf{Alg}_{\mathbb{K}}$ determined by the abelian algebras is isomorphic to the category $\mathsf{Vect}_{\mathbb{K}}$ of vector spaces over $\mathbb{K}$. An algebra A is abelian if and only if $+\colon A \times A \to A$ is an algebra morphism, which makes $(A, +, 0)$ an internal abelian group, so an **abelian object** in the sense of [2].

11.2.2 Algebras over infinite fields

We assume that the field $\mathbb{K}$ is infinite, so that we can use the following result (Theorem 11.2.1, which is Corollary 2 on page 8 of [29]). We first fix some terminology. For a given set S, a **polynomial** with variables in S is an element of the free $\mathbb{K}$-algebra on S. Recall that the left adjoint $\mathsf{Set} \to \mathsf{Alg}_{\mathbb{K}}$ factors as a composite of the *free magma* functor $M\colon \mathsf{Set} \to \mathsf{Mag}$ with the *magma algebra* functor $\mathbb{K}[-]\colon \mathsf{Mag} \to \mathsf{Alg}_{\mathbb{K}}$. The elements of $M(S)$ are non-associative words in the alphabet S, and the elements of $\mathbb{K}[M(S)]$, the polynomials, are $\mathbb{K}$-linear combinations of such words. A **monomial** in $\mathbb{K}[M(S)]$ is any scalar multiple of an element of $M(S)$. The **type** of a monomial $\varphi(x_1, \ldots, x_n)$ is the element $(k_1, \ldots, k_n) \in \mathbb{N}^n$ where k_i is the degree of x_i in $\varphi(x_1, \ldots, x_n)$. A polynomial is **homogeneous** if its monomials are all of the same type. Any polynomial may thus be written as a sum of homogeneous polynomials, which are called its **homogeneous components**.

Theorem 11.2.1. *[29] If $\mathcal{V}$ is a variety of algebras over an infinite field, then all of its laws are of the form $\phi(x_1, \ldots, x_n) = 0$, where $\phi(x_1, \ldots, x_n)$ is a (non-associative) polynomial, each of whose homogeneous components $\psi(x_{i_1}, \ldots, x_{i_m})$ again gives rise to a law $\psi(x_{i_1}, \ldots, x_{i_m}) = 0$ in $\mathcal{V}$.* □

11.2.3 Description of $B\flat X$ in $\mathsf{Alg}_{\mathbb{K}}$

Let B and X be free $\mathbb{K}$-algebras. Then the object $B\flat X$, being the kernel of the morphism $(1_B\ 0)\colon B + X \to B$, consists of those polynomials with variables in B and in X which can be written in a form where all of their monomials contain variables in X. For instance, given b, $b' \in B$ and $x \in X$, the expression $(b(xx))b'$ is allowed, but b or b' are not.

11.2.4 The reflection to a subvariety $\mathcal{V}$ of $\mathsf{Alg}_{\mathbb{K}}$

Let B and X be free $\mathbb{K}$-algebras. We write $\overline{B}$ and $\overline{X}$ for their respective reflections into $\mathcal{V}$, which are free $\mathcal{V}$-algebras. These induce short exact sequences in $\mathsf{Alg}_{\mathbb{K}}$ such as

$$0 \longrightarrow [X] \longrightarrow X \xrightarrow{\eta_X} \overline{X} \longrightarrow 0$$

where η_X is the unit at X of the reflection from $\mathsf{Alt}_{\mathbb{K}}$ to $\mathcal{V}$. We never write the right adjoint inclusion, but note that it preserves all limits. The kernel $[X]$ is a kind of relative commutator; all of its elements are laws of $\mathcal{V}$. Reflecting sums now, then taking kernels to the left, we obtain horizontal split exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (B\flat X)\cap[B+X] & \longrightarrow & [B+X] & \rightleftarrows & [B] & \longrightarrow & 0\\
 & & \vdots & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B\flat X & \longrightarrow & B+X & \overset{(1_B\ 0)}{\rightleftarrows} & B & \longrightarrow & 0\\
 & & \vdots{\scriptstyle \rho_{B,X}} & & \downarrow{\scriptstyle \eta_{B+X}} & & \downarrow{\scriptstyle \eta_B} & & \\
0 & \longrightarrow & \overline{B}\flat_{\mathcal{V}}\overline{X} & \longrightarrow & \overline{B}+_{\mathcal{V}}\overline{X} & \overset{(1_{\overline{B}}\ 0)}{\rightleftarrows} & \overline{B} & \longrightarrow & 0
\end{array}$$

in $\mathsf{Alg}_{\mathbb{K}}$, where the sum $\overline{B}+_{\mathcal{V}}\overline{X} \cong \overline{B+X}$ is taken in $\mathcal{V}$. Using, for instance, the 3×3 Lemma, it is not difficult to see that the induced dotted arrow $\rho_{B,X}$ is a surjective algebra homomorphism. In fact, the upper left square is a pullback, and we have three vertical short exact sequences. The one on the left allows us to view the elements of $\overline{B}\flat_{\mathcal{V}}\overline{X}$ as polynomials in $B\flat X$, modulo those laws which hold in $\mathcal{V}$ that are expressible in $B\flat X$. These laws are precisely the elements of the top left intersection. We freely use this interpretation in what follows, abusing terminology and notation by making no distinction between the equivalence class of polynomials that is an element in the quotient $\overline{B}\flat_{\mathcal{V}}\overline{X}$, and an element in $B\flat X$ which represents it.

11.2.5 Subvarieties of $\mathsf{Lie}_{\mathbb{K}}$ need not be (LACC)

Subvarieties of locally algebraically cartesian closed categories need no longer be such: we may take the variety of Lie algebras that satisfy $x(yz) = 0$ as an example.

Proposition 11.2.2. *Let $\mathcal{V}$ be a variety of non-associative algebras in which $x(yz) = 0$ is a law. If $\mathcal{V}$ is locally algebraically cartesian closed, then it is abelian.*

Proof. Let B, X and Y be free algebras in $\mathcal{V}$, respectively generated by their elements b, x and y. Then the split epimorphisms

$$((1_B\ 0)\colon B + X \to B, \quad \iota_B\colon B \to B + X)$$

and

$$((1_B\ 0)\colon B + Y \to B, \quad \iota_B\colon B \to B + Y)$$

correspond to the free B-actions respectively generated by x and y. Their sum in $\mathsf{Pt}_B(\mathcal{V})$ is

$$((1_B\ 0\ 0)\colon B + X + Y \to B, \quad \iota_B\colon B \to B + X + Y).$$

Applying the kernel functor, (LACC) tells us that the canonical morphism

$$(B\flat\iota_X\ B\flat\iota_Y)\colon B\flat X + B\flat Y \to B\flat(X + Y)$$

is an isomorphism (Theorem 11.1.1). When considering the sum $B\flat X + B\flat Y$ as a subobject of the coproduct $B + X + B + Y$, we write b_1 and b_2 for the generators of the two distinct copies of B; then $(B\flat\iota_X\ B\flat\iota_Y)$ maps the b_i to b, sends x to x and y to y.

Now $x \in B\flat X$ and $yb_2 \in B\flat Y$ are such that $x \cdot yb_2$ is sent to zero by the above isomorphism, since the law $x(yz) = 0$ holds in $\mathcal{V}$. As a consequence,

$x \cdot yb_2$ is zero in the sum $B\flat X + B\flat Y$. Recall that $b_2 \notin B\flat Y$, so that yb_2 cannot be decomposed as a product of y and b_2 in $B\flat Y$. By Theorem 11.2.1, yb_2 can also not be written as a product in which more than one y or b_2 appears, unless yb_2 is zero in $B\flat Y$. As a consequence, $x \cdot yb_2$ can only be zero if either yb_2 is zero in $B\flat Y$, or $xz = 0$ is a law in $\mathcal{V}$. In the former case, yb_2 is zero in the sum $B + Y$, which is a free algebra on $\{b_2, y\}$; then $yz = 0$ is a law in $\mathcal{V}$. In either case, $\mathcal{V}$ is abelian. □

11.2.6 (Anti)associative algebras

Essentially the same argument gives us two further examples, which we shall need later on:

Proposition 11.2.3. *If a variety of either associative or antiassociative algebras is locally algebraically cartesian closed, then it is abelian.*

Proof. In the antiassociative case we have x, $-xb_1 \in B\flat X$ and b_2y, $y \in B\flat Y$ such that $x \cdot b_2y$ and $-xb_1 \cdot y$ are sent to the same element in $B\flat(X + Y)$ by the above isomorphism $(B\flat\iota_X\ B\flat\iota_Y)\colon B\flat X + B\flat Y \to B\flat(X + Y)$.

Similarly, in the associative case, we see that $xb_1 \cdot y$ and $x \cdot b_2y$ are two distinct elements of the sum $B\flat X + B\flat Y$ which the morphism $(B\flat\iota_X\ B\flat\iota_Y)$ sends to one and the same element of $B\flat(X + Y)$. □

Lemma 11.2.4. *Any variety of* $\mathbb{K}$*-algebras that satisfies the law* $x(xy) = 0$ *is a subvariety of* $\mathsf{AAAlg}_{\mathbb{K}}$.

Proof. Taking $x = a + b$ and $y = c$ gives us

$$0 = (a + b)((a + b)c) = a(ac) + b(ac) + a(bc) + b(bc) = b(ac) + a(bc)$$

so that $a(bc) = -b(ac)$. It follows that $(uv)w = -w(uv) = u(wv) = -u(vw)$ is a law in $\mathcal{V}$, and $\mathcal{V}$ is a variety of antiassociative algebras. □

11.2.7 Algebraic coherence

Theorem 11.2.1 gives us a characterisation of algebraic coherence for varieties of $\mathbb{K}$-algebras.

Theorem 11.2.5. *Let* $\mathbb{K}$ *be an infinite field. If* $\mathcal{V}$ *is a variety of non-associative* $\mathbb{K}$*-algebras, then the following are equivalent:*

(i) $\mathcal{V}$ *is algebraically coherent;*

(ii) there exist $\lambda_1, \ldots, \lambda_{16}$ *in* $\mathbb{K}$ *such that*

$$z(xy) = \lambda_1 y(zx) + \lambda_2 x(yz) + \lambda_3 y(xz) + \lambda_4 x(zy) + \lambda_5 (zx)y + \lambda_6 (yz)x + \lambda_7 (xz)y + \lambda_8 (zy)x$$

$$(xy)z = \lambda_9 y(zx) + \lambda_{10} x(yz) + \lambda_{11} y(xz) + \lambda_{12} x(zy) + \lambda_{13} (zx)y + \lambda_{14} (yz)x + \lambda_{15} (xz)y + \lambda_{16} (zy)x$$

are laws in $\mathcal{V}$*;*

(iii) $\mathcal{V}$ *is an* Orzech category of interest *[28].*

Proof. From the results of [10] we already know that (iii) implies (i). It follows immediately from the definition of an *Orzech category of interest* that (ii) implies (iii). To see that (i) implies (ii), we take free B-actions as in the first part of the proof of Proposition 11.2.2 and obtain the regular epimorphism

$$(B\flat\iota_X \; B\flat\iota_Y) \colon B\flat X + B\flat Y \to B\flat(X+Y).$$

Any element $b(xy)$ of $B\flat(X+Y)$ is the image through this morphism of some polynomial $\psi(b_1, x, b_2, y)$ in $B\flat X + B\flat Y$. Note that this polynomial cannot contain any monomials obtained as a product of a b_i with xy or yx. This allows us to write, in the sum $B + X + Y$, the element $b(xy)$ as

$$\lambda_1 y(bx) + \lambda_2 x(yb) + \lambda_3 y(xb) + \lambda_4 x(by) + \lambda_5 (bx)y + \lambda_6 (yb)x + \lambda_7 (xb)y + \lambda_8 (by)x + \nu\phi(b, x, y)$$

for some $\lambda_1, \ldots, \lambda_8, \nu \in \mathbb{K}$, where $\phi(b, x, y)$ is the part of the polynomial in b, x and y which is not in the homogeneous component of $b(xy)$. Since $B+X+Y$ is the free $\mathcal{V}$-algebra on three generators b, x and y, from Theorem 11.2.1 we deduce that the first equation in (ii) is again a law in $\mathcal{V}$. Analogously for $(xy)b$ we deduce the second equation in (ii). □

Remark 11.2.6. This result may be used to prove the claim made in [10] that the category of **Jordan algebras**—commutative and such that $(xy)(xx) = x(y(xx))$—is not algebraically coherent. Indeed, as explained in [27], it is not a *category of interest*.

In the case of alternating algebras, this characterisation becomes more precise:

Theorem 11.2.7. *Let $\mathbb{K}$ be an infinite field. If $\mathcal{V}$ is a subvariety of $\mathsf{Alt}_{\mathbb{K}}$, then the following are equivalent:*

(i) $\mathcal{V}$ is algebraically coherent;

(ii) $\mathcal{V}$ is a subvariety of either $\mathsf{AAAlg}_{\mathbb{K}}$ or $\mathsf{Lie}_{\mathbb{K}}$.

Proof. (ii) implies (i) since $\mathsf{AAAAlg}_{\mathbb{K}}$ and $\mathsf{Lie}_{\mathbb{K}}$ are *Orzech categories of interest* [28], so their subvarieties are algebraically coherent. To see that (i) implies (ii), we first use anticommutativity to simplify the law given in Theorem 11.2.5 to

$$z(xy) = \lambda y(zx) + \mu x(yz)$$

for some λ and μ in $\mathbb{K}$. Choosing, in turn, $y = z$ and $x = z$, we see that

1. either $\lambda = -1$ or $z \cdot zx = 0$ is a law in $\mathcal{V}$, and
2. either $\mu = -1$ or $x \cdot xy = 0$ is a law in $\mathcal{V}$.

In any of the latter cases, $\mathcal{V}$ is a variety of antiassociative algebras by Lemma 11.2.4, which makes it abelian (Proposition 11.2.3). We are left with the situation when $\lambda = \mu = -1$, which means that the Jacobi identity is a law in $\mathcal{V}$, so that $\mathcal{V}$ a variety of Lie algebras. □

Example 11.2.8. The variety of *alternating associative algebras* is an example. We have that $0 = x(yy) = (xy)y$ is a law, so that by Lemma 11.2.4 those algebras are antiassociative as well. It follows that $xyz = 0$ is a law. We regain a variety as in Proposition 11.2.2, so since it is not abelian, it cannot be (LACC).

11.2.8 A characterisation of Lie algebras amongst alternating algebras

The condition (LACC) eliminates one of the two options in Theorem 11.2.7.

Theorem 11.2.9. *Let $\mathbb{K}$ be an infinite field. If $\mathcal{V}$ is a locally algebraically cartesian closed variety of alternating $\mathbb{K}$-algebras, then it is a subvariety of $\mathsf{Lie}_{\mathbb{K}}$. In other words, $\mathsf{Lie}_{\mathbb{K}}$ is the largest* (LACC) *variety of alternating $\mathbb{K}$-algebras. Thus for any variety $\mathcal{V}$ of alternating $\mathbb{K}$-algebras, the following are equivalent:*

(i) $\mathcal{V}$ *is a subvariety of a* (LACC) *variety of alternating* $\mathbb{K}$*-algebras;*

(ii) the Jacobi identity is a law in $\mathcal{V}$.

Proof. This combines Theorem 11.2.7 with Proposition 11.2.3. □

Remark 11.2.10. By Proposition 11.2.2, the condition

(iii) $\mathcal{V}$ is (LACC)

is strictly stronger than the equivalent conditions (i) and (ii).

Remark 11.2.11. We do not know any non-abelian examples of (LACC) subvarieties of $\mathsf{Lie}_{\mathbb{K}}$, or whether such subvarieties even exist.

11.3 Non-alternating algebras

An important question which we have to leave open for now, is what happens when the algebras we consider are not alternating. We end this note with some of our preliminary findings.

Some of the results and techniques used in the previous section are valid for non-alternating algebras of course. For instance, Proposition 11.2.2, Proposition 11.2.3 and Theorem 11.2.5 are.

Proposition 11.2.2 contradicts—in spirit only, since we are working over a field—Proposition 6.9 in [17], which claims that the category of all commutative non-unitary rings satisfying $xyz = 0$ is locally algebraically cartesian closed. It turns out that the argument given in Proposition 11.2.2 is still valid in this case, and shows that the variety under consideration, should it be (LACC), would be abelian—which is false.

We noticed that the functor R constructed in the proof of [17, Proposition 6.9] is not well defined on morphisms. Let us give a concrete example showing this in detail. We follow the notations from [17, Proposition 6.9]. Let $B = \langle b\rangle = \mathbb{Z}[b]$ act on the commutative ring

$$X = \langle x, y \mid xx = xy = yy = 0\rangle = \mathbb{Z}[x, y]/(xx, xy, yy)$$

by $bx = y$ and $by = 0$. Consider

$$M = \langle m, p, q \mid mp = q,\ mq = pq = mm = pp = qq = 0\rangle$$

with the trivial B-action, and let $g\colon X \to M$ be the ring homomorphism sending x and y to m. Let $f \in R(X)$ be defined by $f(n,b) = nx + bx$. Then

$$R(g)(f)(0,b)\cdot p = (g\circ f)(0,b)\cdot p = g(y)\cdot n = mp = q \neq 0,$$

which shows that $R(g)(f)$ is not an element of $R(M)$.

11.3.1 Leibniz algebras

The category $\mathsf{Leib}_{\mathbb{K}}$ of **(right) Leibniz algebras** [23] over $\mathbb{K}$ is the subvariety of $\mathsf{Alg}_{\mathbb{K}}$ satisfying the **(right) Leibniz identity** $(xy)z = x(yz) + (xz)y$. This law is clearly equivalent to the Jacobi identity when the algebras are alternating, so that a Lie algebra is the same thing as an alternating Leibniz algebra. However, examples of non-alternating Leibniz algebras exist. Analogously, we can consider the category of **(left) Leibniz algebras**, with corresponding identity $x(yz) = (xy)z + y(xz)$. Both categories are of course equivalent.

We do not know whether Theorem 11.2.9 extends to the non-alternating case. What is certain, though, is that the category of Leibniz algebras is not locally algebraically cartesian closed. Indeed, using the notations of Proposition 11.2.2, the Leibniz identity allows us to deduce

$$\begin{cases}(xy)b = x(yb) + (xb)y\\ (xy)b = -x(by) + (xb)y\end{cases}$$

so that $x\cdot yb = -x\cdot by$ in $B + X + Y$. This means that $x\cdot yb_2$ and $-x\cdot b_2 y$ are two distinct elements of $B\flat X + B\flat Y$ which are sent to the same element of $B\flat(X+Y)$ by the morphism $(B\flat\iota_X\ B\flat\iota_Y)$. Hence this morphism cannot be an isomorphism, and $\mathsf{Leib}_{\mathbb{K}}$ is not (LACC).

We may ask ourselves what happens in the "intersection" between right and left Leibniz algebras. They are called **symmetric Leibniz algebras** and, as shown in [26], the chain of inclusions $\mathsf{Lie}_{\mathbb{K}} \subseteq \mathsf{SLeib}_{\mathbb{K}} \subseteq \mathsf{Leib}_{\mathbb{K}}$ is strict. Doing a rearrangement of terms as in

$$b(xy) = (bx)y + x(by) = b(xy) + (by)x + x(by),$$

we see that $(by)x + x(by) = 0$. From this we may conclude that, in order to be (LACC), a variety of symmetric Leibniz algebras must either be alternating or abelian. We thus regain the known cases of Lie algebras and vector spaces.

11.3.2 Free algebra on one generator

A variety of algebras is alternating precisely when the free algebra on a single generator is abelian: $xx = 0$ is a law in the variety, if and only if the bracket vanishes on the algebra freely generated by $\{x\}$. This corresponds to the condition that *the free algebra on a single generator admits an internal abelian group structure*. This condition makes sense in arbitrary semi-abelian varieties, and we may ask ourselves whether perhaps it is implied by (LACC), as in the case of symmetric Leibniz algebras. This would allow us to drop the condition that $\mathcal{V}$ is alternating in Theorem 11.2.9.

The example of crossed modules proves that this is false. In [9] it is shown that on the one hand, a crossed module $\partial\colon T \to G$ with action ξ admits an internal abelian group structure if and only if the groups T and G are abelian and the action ξ is trivial. On the other hand, the free crossed module on a single generator is the inclusion $\kappa_{\mathbb{Z},\mathbb{Z}}\colon \mathbb{Z}\flat\mathbb{Z} \to \mathbb{Z} + \mathbb{Z}$, equipped with the conjugation action. We see that in this case the free object on one generator is not abelian, even though XMod is a locally algebraically cartesian closed semi-abelian variety. However, it is not a variety of non-associative algebras of course.

Perhaps this is not the right conceptualisation, and we must think of other ways of making the law $xx = 0$ categorical. The question then becomes whether (LACC), or any other appropriate categorical-algebraic condition, would imply this new characterisation.

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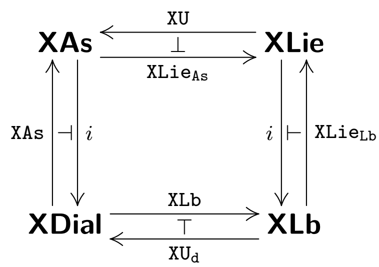
Chapter 12

Future research

At the moment of writing this dissertation, several ongoing joint projects are being worked on. One such with José Manuel Casas and Tim Van der Linden is to study low-dimensional homology of *multiplicative Hom-Lie algebras* [7]. This category is of special interest since, even though it has some good algebraic behaviour, it is an example of semi-abelian category where the *Smith is Huq* condition [10], *normality of Higgins commutators* [2] or even the *universal central extension* condition [1] do not hold.

A joint project with Rafael Fernández-Casado is to generalise the notion of actor of crossed modules of Leibniz algebras to the *dialgebras* setting [8]. The aim is to extend Chapter 6 to complete the square-shaped diagram formed by associative algebras, Lie algebras, dialgebras and Leibniz algebras:



Note that the first two cases were studied in my coauthor's Ph.D. thesis [4].

An other ongoing project with James Gray is to prove that all categories of Lie objects over an abelian, cocomplete, symmetric, closed, monoidal category are (LACC) [5], generalizing his proof in the case of R-modules [6]. This result is interesting since it will add some examples to the short list of known (LACC) categories, as *Lie superalgebras* or *differential graded Lie algebras*. Moreover, it will automatically imply that Lie algebras in the *Loday-Pirashvili* category [9] are (LACC). Thus we regain the non-(LACC) category of Leibniz algebras as a full reflective subcategory of a (LACC) one.

Of the lines of work that can emerge from this thesis, maybe the most ambitious one is to extend Chapter 11. The obvious question is what happens if we consider all non-associative algebras instead of alternating ones. The author believes that the characterisation is still valid, but no proof has been found yet. In any case, a negative solution, i.e. a counterexample, would also be a very meaningful and surprising result.

In Chapter 10, some questions are left open. Since $(\mathsf{BiAlg}_{\mathbb{K},c})^{\mathrm{op}} \simeq \mathsf{Mon}\big((\mathsf{Alg}_{\mathbb{K},c})^{\mathrm{op}}\big)$ and $(\mathsf{Hopf}_{\mathbb{K},c})^{\mathrm{op}} \simeq \mathsf{Gp}\big((\mathsf{Alg}_{\mathbb{K},c})^{\mathrm{op}}\big)$, Theorem 10.2.4 implies that the category of commutative Hopf algebras is *coprotomodular*. Nevertheless, it is not known if it is an example of a *co-semi-abelian* category.

A group G is called *capable* if there exists another group Q such that $G \cong Q/\mathrm{Z}(Q)$. In [3] it was proven that a group is capable if and only if the *exterior centre* $Z^{\wedge}(G) = \{h \in G \mid h \wedge g = 1 \text{ for all } g \in G\}$ is trivial. This was extended to Lie algebras in [11] and it is interesting to see if it is still true in the Leibniz algebras case, since in Chapter 5 the non-abelian exterior product of Leibniz algebras was introduced.

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Resumo da Tese de doutoramento (in Galician)

Categorical-algebraic methods in non-commutative and non-associative algebra

Resumo abreviado:
Métodos categórico-alxébricos en álxebra non asociativa e non conmutativa

Esta tese ten un dobre obxectivo: o primeiro é empregar métodos categóricos e alxébricos para estudar propiedades homolóxicas dalgunhas estruturas alxébricas variantes de Lie ou Leibniz, e o segundo consiste en empregar métodos categóricos e alxébricos para estudar e caracterizar certas coñecidas estruturas. Por unha parte, estúdase a teoría de extensións centrais universais e o produto tensor non abeliano, calcúlanse explicitamente algúns grupos de homoloxía, e resólvense tamén problemas sobre álxebras envolventes universais e accións. Por outra parte, centrarémonos en dar caracterizacións categóricas de certas estruturas alxébricas, como caracterizar a categoría de grupos dentro da categoría de monoides, a categoría de álxebras de Hopf dentro das biálxebras coconmutativas ou a categoría de álxebras de Lie dentro das álxebras non asociativas alternadas.

Dende que nos anos 50 se definiron as categorías abelianas como unha abstracción categórica das propiedades dos grupos abelianos e módulos, tratouse de atopar un marco similar que sexa capaz de abstraer as propiedades das categorías de grupos (non necesariamente abelianos), aneis ou álxebras.

Houbo varios intentos de conseguir este nivel de abstracción ao longo dos anos: salientando os esforzos de Higgins [26], Huq [28] ou R.-Grandjeán [35], pódese consultar en [29] unha lista máis ampla. Xa que ningún destes intentos foi plenamente satisfactorio e as conexións entre eles non estaban moi claras, non se estudaron con moita intensidade e nin sequera se lles deu un nome. En 1999, Janelidze, Márki e Tholen decatáronse de que a propiedade de *Barr-exactitude* [3] xunto co concepto de *Bourn-protomodularidade* [5] proporcionan un contexto que simplifica e unifica os "antigos" sistemas de axiomas mencionados previamente; e a través deles, pódense explorar facilmente as relacións coa álxebra categórica moderna.

Expresada nos termos dos axiomas "novos", unha *categoría semi-abeliana* é unha categoría punteada que é *Barr-exacta* e *Bourn-protomodular* con sumas finitas. En particular, podemos atopar moitas das estruturas alxébricas non asociativas e non conmutativas estudadas na literatura [4], incluíndo aquelas cunha estrutura de grupo subxacente. Precisamente, calquera variedade de álxebras que ten entre as súas operacións e identidades as de teoría de grupos, é semi-abeliana.

Unha das vantaxes deste marco categórico é que permite un estudo unificado de moitas propiedades homolóxicas. Por exemplo, nunha categoría semi-abeliana cúmprense os lemas diagramáticos clásicos (o *Lema corto escindido dos cinco*, o *Lema* 3×3, o *Lema da serpente*, os *Teoremas de Isomorfía de Noether*). Como se pode ver en [37], a teoría das categorías semi-abelianas está perfectamente equipada para o estudo da (co)homoloxía non abeliana e a correspondente teoría de homotopía, unificando moitos aspectos básicos das teorías de (co)homoloxía clásicas de grupos, álxebras de Lie e módulos cruzados.

Dende un punto de vista alxébrico, a teoría de cohomoloxía de álxebras de Lie foi definida en [11], tratando de dar una construción alxébrica da cohomoloxía de espazos topolóxicos de grupos de Lie compactos. Esta teoría estudouse ao longo dos anos e estendeuse a moitas estruturas como *módulos cruzados de álxebras de Lie* [8, 7], *superálxebras de Lie* [33], *álxebras de Lie-Rinehart* [27, 36], *(super)álxebras de Leibniz* [30], *n-álxebras de Lie* [1], *n-álxebras de Leibniz* [9], etc.

A teoría de álxebras non asociativas está fortemente relacionada con áreas das matemáticas moi diferentes e ten moitas aplicacións en física, mecánica, bioloxía e outras ciencias. Como insignias de álxebras non asociativas, atopamos as álxebras de Lie e de Jordan, que tiveron unha relevancia enorme durante o decurso do século pasado. O estudo das álxebras non asociativas engloba a teoría de R-álxebras non necesariamente asociativas (sendo as álxebras asociativas un caso especial moi importante), onde R pode ser un anel ou un corpo. Os problemas que se abordan son variados, como por exemplo o estudo de solubilidade ou nilpotencia, clasificacións, caracterizacións, relacións coa xeometría diferencial, etc.

O obxectivo desta tese é dobre: o primeiro é empregar métodos categóricos-alxébricos para estudar propiedades homolóxicas dalgunhas das estruturas non asociativas, semi-abelianas, xa mencionadas; o segundo é usar métodos categórico-alxébricos para estudar propiedades categóricas e dar caracterizacións categóricas dalgunhas coñecidas estruturas alxébricas.

Por unha parte, estudaranse as extensións centrais universais xunto co produto tensor non abeliano, e empregaranse para calcular explicitamente algúns grupos de homoloxía [10, 13, 19, 21, 20], e resolveranse algúns problemas sobre álxebras envolventes universais e accións [16, 17, 6, 22]. Por outra banda, centrarémonos en dar caracterizacións categóricas dalgunhas estruturas alxébricas, como unha caracterización da categoría de grupos dentro dos monoides [18], de álxebras de Hopf coconmutativas dentro das biálxebras coconmutativas [24] e das álxebras de Lie dentro das álxebras alternadas [23].

A continuación darase unha explicación detallada de cada un dos capítulos, e salientándose en cada un os resultados máis importantes. No Capítulo 1 estúdase as extensións centrais universais na categoría de álxebras de Lie-Rinehart, dando unha descrición alxébrica. Tamén se estuda o levantamento de automorfismos e derivacións de extensións centrais. Despois, seguindo o modelo de Ellis de produto tensor non abeliano [15], dáse unha definición do produto tensor non abeliano de álxebras de Lie-Rinehart, relacionándoo coas extensións centrais universais.

Teorema 1.3.11. *Sexa L unha álxebra de Lie-Rinehart perfecta. Entón*

$$\mathsf{Ker}\,\mathfrak{u} \longrightarrow \mathfrak{uce}_{\mathrm{A}} L \xrightarrow{\mathfrak{u}} L,$$

é una extensión central universal de L. Ademais, se L non ten centro, entón $\mathsf{Ker}\,\mathfrak{u} = \mathsf{Z}_{\mathrm{A}}(\mathfrak{uce}_{\mathrm{A}} L)$.

Proposición 1.5.9. *Dada unha álxebra de Lie-Rinehart perfecta L, o produto tensor non abeliano* $L \otimes L$ *é a extensión central universal de L, onde a quasi-acción de L sobre si mesmo é o corchete de Lie.*

No Capítulo 2 dáse unha definición do produto tensor non abeliano no caso de superálxebras de Lie, estudando diversas propiedades e emprégase para o estudo de extensións centrais universais. Preséntase unha definición da homoloxía non abeliana de dimensións baixas e relaciónase coa homoloxía cíclica de superálxebras. Para rematar, defínese un produto exterior non abeliano probando un análogo do teorema de Miller, una fórmula de Hopf e unha sucesión exacta de seis termos na homoloxía de superálxebras.

Teorema 2.5.9. *Sexa A unha superálxebra asociativa con unidade. Entón existe una sucesión exacta de supermódulos:*

$$\frac{A}{[A,A]} \otimes_{\mathbb{K}} \mathrm{HC}_1(A) \longrightarrow \mathcal{H}_1\big(A, \mathrm{V}(A)\big) \longrightarrow \mathcal{H}_1(A,[A,A]) \longrightarrow \mathrm{HC}_1(A) \longrightarrow \mathrm{HC}_1^M(A) \longrightarrow \frac{[A,A]}{\big[A,[A,A]\big]} \longrightarrow 0.$$

Teorema 2.6.6. *Sexa M un ideal graduado dunha superálxebra de Lie P. Entón existe una sucesión exacta*

$$\mathsf{Ker}(P \wedge M \to P) \to \mathrm{H}_2(P) \to \mathrm{H}_2(P/M) \to \frac{M}{[P,M]} \to \mathrm{H}_1(P) \to \mathrm{H}_1(P/M) \to 0.$$

No Capítulo 3 atópase a extensión central universal das superálxebras de matrices $\mathfrak{sl}(m,n,A)$ onde A é unha superálxebra asociativa e $m+n = 3,4$, relacionándoa coa superálxebra de Steinberg $\mathfrak{st}(m,n,A)$. Calcúlanse $\mathrm{H}_2\left(\mathfrak{sl}(m,n,A)\right)$ e $\mathrm{H}_2\left(\mathfrak{st}(m,n,A)\right)$, e para rematar, introdúcese un novo método usando o produto tensor non abeliano de superálxebras de Lie (definido no capítulo anterior) para atopar unha conexión entre $\mathrm{H}_2\left(\mathfrak{sl}(m,n,A)\right)$ e a homoloxía cíclica de superálxebras asociativas cando $m+n \geqslant 3$.

Teorema 3.8.1. *Sexa K un anel conmutativo con unidade e A unha K-superálxebra asociativa con unidade. Entón,*

$$\mathrm{H}_2\left(\mathfrak{st}(m,n,A)\right) = \begin{cases} 0 & \textit{for } m+n \geqslant 5 \textit{ or } m=2, n=1, \\ A_3^6 & \textit{for } m=3, n=0, \\ A_2^6 & \textit{for } m=4, n=0, \\ \Pi(A_2)^6 & \textit{for } m=3, n=1, \\ A_2^4 \oplus A_0^2 & \textit{for } m=2, n=2, \end{cases}$$

onde A_m é o cociente de A polo ideal $mA + A[A,A]$ (Definición 3.2.3) e Π é o funtor cambio de paridade.

Teorema 3.8.2. *Sexa K un anel conmutativo con unidade e A unha K-superálxebra asociativa con unidade con unha K-base que conteña a* *unidade. Entón,*

$$\mathrm{H}_2\big(\mathfrak{sl}(m,n,A)\big) = \begin{cases} \mathrm{HC}_1(A) & \textit{for } m+n \geqslant 5 \textit{ or } m=2, n=1, \\ \mathrm{HC}_1(A) \oplus A_3^6 & \textit{for } m=3, n=0, \\ \mathrm{HC}_1(A) \oplus A_2^6 & \textit{for } m=4, n=0, \\ \mathrm{HC}_1(A) \oplus \Pi(A_2)^6 & \textit{for } m=3, n=1, \\ \mathrm{HC}_1(A) \oplus A_2^4 \oplus A_0^2 & \textit{for } m=2, n=2, \end{cases}$$

onde A_m é o cociente de A polo ideal $mA + A[A,A]$ (Definición 3.2.3) e Π *é o funtor cambio de paridade.*

Seguindo as liñas do capítulo anterior, no Capítulo 4 complétase o problema de atopar a extensión central universal na categoría de superálxebras de Leibniz das superálxebras de matrices $\mathfrak{sl}(m,n,D)$ cando $m+n \geqslant 3$ e D é unha superdiálxebra, resolvendo o problema particular de cando se trata dunha álxebra asociativa, superálxebra ou diálxebra. Para completar esta tarefa, empregamos un método orixinal distinto do habitual que se pode atopar na literatura. Tamén se introduce o cadrado tensor non abeliano de álxebras de Leibniz para estudar as súas relacións coa extensión central universal.

Teorema 4.6.1. *Sexa R un anel conmutativo con unidade e D unha R-superdiálxebra asociativa con unidade e unha R-base que conteña a bar-unidade. Entón,*

$$\mathrm{HL}_2\big(\mathfrak{sl}(m,n,D)\big) = \begin{cases} \mathrm{HHS}_1(D) & \textit{for } m+n \geqslant 5 \textit{ or } m=2, n=1, \\ \mathrm{HHS}_1(D) \oplus D_3^6 & \textit{for } m=3, n=0, \\ \mathrm{HHS}_1(D) \oplus D_2^6 & \textit{for } m=4, n=0, \\ \mathrm{HHS}_1(D) \oplus \Pi(D_2)^6 & \textit{for } m=3, n=1, \\ \mathrm{HHS}_1(D) \oplus D_2^4 \oplus D_0^2 & \textit{for } m=2, n=2, \end{cases}$$

onde D_m e o resultado de cocientar D polo ideal $mD + ([D,D] \dashv D)$ (Definición 4.4.2) e Π *é o funtor cambio de paridade.*

Teorema 4.6.2. *Sexa R un anel conmutativo con unidade e D unha R-superdiálxebra asociativa con unidade e unha R-base que conteña a bar-*

unidade. Entón,

$$\mathrm{HL}_2\left(\mathfrak{stl}(m,n,D)\right) = \begin{cases} 0 & \textit{for } m+n \geqslant 5 \textit{ or } m=2, n=1, \\ D_3^6 & \textit{for } m=3, n=0, \\ D_2^6 & \textit{for } m=4, n=0, \\ \Pi(D_2)^6 & \textit{for } m=3, n=1, \\ D_2^4 \oplus D_0^2 & \textit{for } m=2, n=2, \end{cases}$$

onde D_m e o resultado de cocientar D polo ideal $mD + ([D,D] \dashv D)$ (Definición 4.4.2) e Π é o funtor cambio de paridade.

No Capítulo 5 introdúcese o produto exterior non abeliano de dous módulos cruzados de álxebras de Leibniz e investígase a súa relación coa homoloxía de Leibniz en dimensión baixa. Este produto aplícase á construción dunha sucesión exacta de oito termos. Por último, establécese unha relación co funtor cuadrático universal, aplicándoo a unha comparación entre a segunda homoloxía de Lie e a segunda homoloxía de Leibniz.

Corolario 5.4.4. *Sexa $0 \to \mathfrak{a} \to \mathfrak{g} \to \mathfrak{h} \to 0$ unha extensión central de álxebras de Leibniz, é dicir, $[a,x] = [x,a] = 0$ para todo $a \in \mathfrak{a}$ e $x \in \mathfrak{g}$. Entón existe a seguinte sucesión exacta:*

$$\begin{aligned} HL_3(\mathfrak{g}) \to HL_3(\mathfrak{h}) \to \operatorname{Coker}\left(\mathfrak{a}\otimes\mathfrak{a} \xrightarrow{\eta} \mathfrak{a}\otimes\frac{\mathfrak{g}}{[\mathfrak{g},\mathfrak{g}]} \oplus \frac{\mathfrak{g}}{[\mathfrak{g},\mathfrak{g}]}\otimes\mathfrak{a}\right) \\ \to HL_2(\mathfrak{g}) \to HL_2(\mathfrak{h}) \to \mathfrak{a} \to HL_1(\mathfrak{g}) \to HL_1(\mathfrak{h}) \to 0, \end{aligned}$$

onde $\eta\colon \mathfrak{a}\otimes\mathfrak{a} \to \mathfrak{a}\otimes\frac{\mathfrak{g}}{[\mathfrak{g},\mathfrak{g}]} \oplus \frac{\mathfrak{g}}{[\mathfrak{g},\mathfrak{g}]}\otimes\mathfrak{a}$ ven dado por $a\otimes b \mapsto (a\otimes\overline{b}, -\overline{a}\otimes b)$, con $\overline{a} = a + [\mathfrak{g},\mathfrak{g}]$ e $\overline{b} = b + [\mathfrak{g},\mathfrak{g}]$ para cada $a, b \in \mathfrak{a}$.

Proposición 5.6.1. *Sexa $\mathfrak{g}$ unha álxebra de Lie. Entón, existe un subespazo vectorial V de $\mathsf{Ker}\{t_{\mathfrak{g}}\colon HL_2(\mathfrak{g}) \to H_2(\mathfrak{g})\}$ tal que temos un epimorfismo $V \to \Gamma(\mathfrak{g}^{\mathrm{ab}})$. De este xeito, se $\mathfrak{g}$ non é unha álxebra de Lie perfecta, $t_{\mathfrak{g}}\colon HL_2(\mathfrak{g}) \to H_2(\mathfrak{g})$ non é un isomorfismo.*

No Capítulo 6 esténdese a noción de biderivación á categoría de módulos cruzados de álxebras de Leibniz a través das accións de álxebras de Leibniz. Isto permítenos construír un obxecto que é o actor baixo certas circunstancias.

Ademais, dáse unha descrición da acción na categoría de módulos cruzados de álxebras de Leibniz en termos de ecuacións. Para rematar, compróbase que baixo certas condicións, o núcleo do morfismo canónico que vai dende un módulo cruzado ao seu actor coincide co seu centro, e introducimos as nocións de biderivacións internas e externas en módulos cruzados.

Teorema 6.4.3. *Sexa* $(\mathfrak{m}, \mathfrak{p}, \eta)$ *e* $(\mathfrak{n}, \mathfrak{q}, \mu)$ *en* ***XLb****. Sempre que se cumpran certas condicións de compatibilidade, existe un homomorfismo de módulos cruzados dende* $(\mathfrak{m}, \mathfrak{p}, \eta)$ *a* $(\mathrm{Bider}(\mathfrak{q}, \mathfrak{n}), \mathrm{Bider}(\mathfrak{n}, \mathfrak{q}, \mu), \Delta)$*. Ademais, o recíproco tamén é certo se estamos nalgún dos seguintes casos:*

$$\mathrm{Ann}(\mathfrak{n}) = 0 = \mathrm{Ann}(\mathfrak{q}), \tag{CON1}$$

$$\mathrm{Ann}(\mathfrak{n}) = 0 \quad e \quad [\mathfrak{q}, \mathfrak{q}] = \mathfrak{q}, \tag{CON2}$$

$$[\mathfrak{n}, \mathfrak{n}] = \mathfrak{n} \quad e \quad [\mathfrak{q}, \mathfrak{q}] = \mathfrak{q}. \tag{CON3}$$

No Capítulo 7 esténdese a módulos cruzados o funtor álxebra envolvente universal entre álxebras asociativas e álxebras de Leibniz. Constrúese un isomorfismo entre a categoría de representacións dun módulo cruzado de álxebras de Leibniz e a categoría de módulos pola esquerda sobre o seu módulo cruzado envolvente universal. O modo de estudar o problema é especialmente interesante xa que o actor na categoría de módulos cruzados de álxebras de Leibniz non existe por norma xeral, así que a proba no caso de Lie non pode ser aplicada. Para rematar, estudamos o devandito funtor dentro do marco da categoría de Loday-Pirashvili [31] para entender este módulo cruzado envolvente universal en termos do caso de módulos cruzados de álxebras de Lie.

Teorema 7.5.1. *A categoría de representacións dun módulo cruzado de álxebras de Leibniz* $(\mathfrak{q}, \mathfrak{p}, \eta)$ *é isomorfa á categoría de módulos pola esquerda sobre o seu módulo cruzado de álxebras envolvente universal* $\mathtt{XUL}(\mathfrak{q}, \mathfrak{p}, \eta)$*.*

No Capítulo 8 investígase en que sentido, para $n \geqslant 3$, as n-álxebras de Lie admiten álxebras envolventes universais. Houbo varios intentos dunha construción (ver [14] e [2]) mais despois de analizalas polo miúdo chegamos á conclusión de que en xeral non son válidas. Para isto, danse contraexemplos e condicións suficientes. Logo, analizamos o problema en toda a súa xeneralidade, demostrando que a universalidade é incompatible co feito de que a categoría de módulos sobre unha n-álxebra de Lie dada sexa equivalente á categoría de módulos sobre a álxebra asociada $\mathrm{U}(L)$. De feito, existe cando menos un *funtor álxebra asociada* $\mathrm{U}\colon n\text{-}\mathsf{Lie}_{\mathbb{K}} \to \mathsf{Alg}_{\mathbb{K}}$ que induce esta equivalencia, pero nunca admite un adxunto pola dereita. Para rematar, defínese unha teoría de (co)homoloxía baseada no funtor álxebra asociada U.

Proposición 8.3.9. $\mathrm{BLb}^{\Lambda}_{n-1}(L) \cong \mathrm{InnDer}(L)$ *se e só se* $\mathcal{K}_{n-1} = \mathcal{W}_{n-1}$.

Teorema 8.4.6. *O funtor* $\mathrm{U}\colon n\text{-}\mathsf{Lie}_{\mathbb{K}} \to \mathsf{Alg}_{\mathbb{K}}$ *ten un adxunto pola dereita se e só se* $n = 2$. *Máis concretamente, para* $n > 2$ *non existe ningún funtor* $F\colon n\text{-}\mathsf{Lie}_{\mathbb{K}} \to \mathsf{Alg}_{\mathbb{K}}$ *cun adxunto pola dereita* $G\colon \mathsf{Alg}_{\mathbb{K}} \to n\text{-}\mathsf{Lie}_{\mathbb{K}}$ *que induza unha equivalencia de categorías entre* $L\text{-}\mathsf{Mod}_{\mathbb{K}}$ *e* $\mathsf{Mod}_{F(L)}$ *para todo* L.

No Capítulo 9 próbase que un monoide M é un grupo se e só se, na categoría de monoides, todos os puntos sobre M son fortes. Este resultado mellora e simplifica o traballo e Montoli, Rodelo e Van der Linden [32] que caracteriza os grupos entre os monoides coma os obxectos protomodulares.

Teorema 9.0.1. *Un monoide* M *é un grupo se e só se, en* Mon*, todos os puntos sobre* M *son fortes.*

No Capítulo 10 dáse unha caracterización universal das álxebras de Hopf entre as biálxebras coconmutativas sobre un corpo alxebricamente pechado: unha biálxebra coconmutativa é unha álxebra de Hopf cando toda extensión escindida sobre ela admite unha descomposición. Tamén se resolve que este resultado non pode ser estendido ao contexto non coconmutativo, probando así que as categorías de biálxebras e álxebras de Hopf non son unitais nin protomodulares.

Teorema 10.3.5. *Se* $\mathbb{K}$ *é un corpo alxebricamente pechado e* Y *é unha biálxebra coconmutativa sobre* $\mathbb{K}$*, entón as seguintes condicións son equivalentes:*

(i) Y *é unha álxebra de Hopf;*

(ii) en $\mathsf{BiAlg}_{\mathbb{K},coc}$*, todas as extensións escindidas sobre* Y *admiten unha descomposición;*

(iii) Y *é un obxecto protomodular en* $\mathsf{BiAlg}_{\mathbb{K},coc}$.

Proposición 10.4.1. *Se* Y *é un obxecto unital de* $\mathsf{BiAlg}_{\mathbb{K}}$*, entón para todo* X *temos un isomorfismo* $X \times Y \cong X \otimes Y$.

No Capítulo 11 próbase que se $\mathbb{K}$ é un corpo infinito, unha variedade de $\mathbb{K}$-álxebras alternadas—non necesariamente asociativas, onde se cumpre $xx = 0$—é localmente alxebricamente cartesiana pechada (segundo a definición de Gray [25]), entón é unha variedade de álxebras de Lie. En particular, $\mathsf{Lie}_{\mathbb{K}}$ é a variedade máis grande. Deste xeito, para unha variedade de $\mathbb{K}$-álxebras alternadas, a identidade de Jacobi convértese nunha condición categórica. Tamén se dá unha caracterización das variedades coherentes alxebricamente (segundo Cigoli, Gray e Van der Linden [12]).

Teorema 11.2.9. *Sexa $\mathbb{K}$ un corpo infinito. Se $\mathcal{V}$ é unha localmente alxebricamente cartesiana pechada de $\mathbb{K}$-álxebras alternadas, entón é unha subvariedade de $\mathsf{Lie}_{\mathbb{K}}$. En outras palabras, $\mathsf{Lie}_{\mathbb{K}}$ é a maior* (LACC) *variedade de $\mathbb{K}$-álxebras alternadas. Entón, para unha variedade de $\mathbb{K}$-álxebras alternadas, son equivalentes:*

(i) $\mathcal{V}$ é unha subvariedade dunha variedade (LACC) *$\mathbb{K}$-álxebras alternadas;*

(ii) a identidade de Jacobi cúmprese en $\mathcal{V}$.

Teorema 11.2.5. *Sexa $\mathbb{K}$ un corpo infinito. Se $\mathcal{V}$ é una variedade de $\mathbb{K}$-álxebras non asociativas, entón son equivalentes:*

(i) $\mathcal{V}$ é coherente alxebricamente;

(ii) existen $\lambda_1, \ldots, \lambda_{16}$ in $\mathbb{K}$ tales que as identidades

$$\begin{aligned} z(xy) = \lambda_1 y(zx) + \lambda_2 x(yz) + \lambda_3 y(xz) + \lambda_4 x(zy) \\ + \lambda_5 (zx)y + \lambda_6 (yz)x + \lambda_7 (xz)y + \lambda_8 (zy)x \end{aligned}$$

$$\begin{aligned} (xy)z = \lambda_9 y(zx) + \lambda_{10} x(yz) + \lambda_{11} y(xz) + \lambda_{12} x(zy) \\ + \lambda_{13} (zx)y + \lambda_{14} (yz)x + \lambda_{15} (xz)y + \lambda_{16} (zy)x \end{aligned}$$

cúmprense en $\mathcal{V}$;

(iii) $\mathcal{V}$ é unha categoría de interese de Orzech *[34].*

Por último, no Capítulo 12 fálase sobre diversos traballos de investigación que o autor está a realizar neste momento, e sobre ideas que poden xurdir tendo esta tese como punto de partida.

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